The adjoint of a matrix:

The transpose matrix of the cofactors of A is called the adioint matrix of A and written as

$$adj(A) = (A_{ij})^t$$

The Inverse of a matrix:

If A is $n \times n$ matrix, then A has inverse if there exists $n \times n$ matrix A^{-1} such that $A^{-1}A = AA^{-1} = \mathrm{I}$, where I is identity matrix.

Definition:

If $|A| \neq 0$ then $A^{-1} = \frac{1}{|A|} adj(A)$

Example:

Find A^{-1} if:

(1) $A = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$

$$A^{-1} = \frac{1}{|A|} adj(A)$$

$$|A| = \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} = 3 - (-2) = 5 \neq 0$$

$$adj(A) = \begin{pmatrix} +1 & -2 \\ -(-1) & +3 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{pmatrix}$$

(2) $A = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{pmatrix}$

$$A^{-1} = \frac{1}{|A|} adj(A)$$

$$|A| = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{vmatrix} = 2(0+1) + 3(-2+1) + 4(1-0) = 2 - 3 + 4 = 3 \neq 0$$

$$adj(A) = \begin{pmatrix} +\begin{vmatrix} 0 & -1 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} -3 & 4 \\ 1 & -2 \end{vmatrix} & +\begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} -3 & 4 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} \end{pmatrix}^t$$

$$= \begin{pmatrix} +(0+1) & -(-2+1) & +(1-0) \\ -(6-4) & +(-4-4) & -(2+3) \\ +(3-0) & -(-2-4) & +(0+3) \end{pmatrix}^t = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -8 & -5 \\ 3 & 6 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & -2 & 3 \\ 1 & -8 & 6 \\ 1 & -5 & 3 \end{pmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 & 3 \\ 1 & -8 & 6 \\ 1 & -5 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & 1 \\ \frac{1}{3} & -\frac{8}{3} & 2 \\ \frac{1}{3} & -\frac{5}{3} & 1 \end{pmatrix}$$

The properties of the adjoint and the inverse:

Let A and B are square matrices then:

(1) $A\,(adjA) = |A|\mathrm{I}$.

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $adj(A) = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$$A\,(adjA) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$|A|\mathrm{I} = -2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\therefore A(adjA) = |A|\mathrm{I}.$$

(2) $adj(AB) = adj(B)adj(A)$.

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 & -1 \\ -4 & -2 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} -3 & -5 \\ -1 & -11 \end{pmatrix}$$

$$adj(AB) = \begin{pmatrix} -11 & 5 \\ 1 & -3 \end{pmatrix}$$

$$adj(A) = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix},\ adj(B) = \begin{pmatrix} -2 & 1 \\ 4 & 5 \end{pmatrix}$$

$$adj(B)adj(A) = \begin{pmatrix} -2 & 1 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -11 & 5 \\ 1 & -3 \end{pmatrix}$$

$$\therefore adj(AB) = adj(B)adj(A)$$

(3) $(A^{-1})^{-1} = A$

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\qquad A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

$$|A^{-1}| = \begin{vmatrix} -2 & 1 \\ -\frac{3}{2} & -\frac{1}{2} \end{vmatrix} = 1 - \frac{3}{2} = -\frac{1}{2}$$

$$(A^{-1})^{-1} = \frac{1}{-\left(\frac{1}{2}\right)}\begin{pmatrix} -\frac{1}{2} & -1 \\ -\frac{3}{2} & -2 \end{pmatrix} = -2\begin{pmatrix} -\frac{1}{2} & -1 \\ -\frac{3}{2} & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = A$$

$$\therefore (A^{-1})^{-1} = A$$

(4) $(A^n)^{-1} = (A^{-1})^n$, $n \equiv$ positive integer

Let $\quad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, n = 2, |A| = -2$

$$(A^2)^{-1} = (A^{-1})^2$$

$$A^2 = AA = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}, \Rightarrow |A^2| = 4$$

$$(A^2)^{-1} = \frac{1}{4}\begin{pmatrix} 22 & -10 \\ -15 & 7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

$$(A^{-1})^2 = \frac{1}{4}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 22 & -10 \\ -15 & 7 \end{pmatrix}$$

$$\therefore (A^n)^{-1} = (A^{-1})^n$$

(5) $(AB)^{-1} = B^{-1}A^{-1}$.

Let $\quad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & -1 \\ -4 & -2 \end{pmatrix}$

$$A^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}, \qquad B^{-1} = -\frac{1}{14}\begin{pmatrix} -2 & 1 \\ 4 & 5 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} -3 & -5 \\ -1 & -11 \end{pmatrix}$$

$$(AB)^{-1} = \frac{1}{28}\begin{pmatrix} -11 & 5 \\ 1 & -3 \end{pmatrix}$$

$$B^{-1}A^{-1} = \left(-\frac{1}{14}\begin{bmatrix} -2 & 1 \\ 4 & 5 \end{bmatrix}\right)\left(-\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\right) = \frac{1}{28}\begin{pmatrix} -11 & 5 \\ 1 & -3 \end{pmatrix}$$

$$\therefore (AB)^{-1} = B^{-1}A^{-1}$$

$$A^{-1}B^{-1} = \left(-\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\right)\left(-\frac{1}{14}\begin{bmatrix} -2 & 1 \\ 4 & 5 \end{bmatrix}\right) = \frac{1}{28}\begin{pmatrix} -16 & -6 \\ 10 & 2 \end{pmatrix}$$

$$A^{-1}B^{-1} \neq (AB)^{-1}$$

(6) $(KA)^{-1} = K^{-1}A^{-1} = \frac{1}{K}A^{-1}$.

Let $\quad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad K = 2$

$$KA = 2\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix} = \begin{pmatrix}2 & 4\\6 & 8\end{pmatrix}, |KA| = -8$$

$$(KA)^{-1} = -\frac{1}{8}\begin{pmatrix}8 & -4\\-6 & 2\end{pmatrix} = \begin{pmatrix}-1 & \frac{1}{2}\\ \frac{3}{4} & -\frac{1}{4}\end{pmatrix}$$

$$\frac{1}{K}A^{-1} = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\begin{bmatrix}4 & -2\\-3 & 1\end{bmatrix}\right) = \left(-\frac{1}{4}\right)\begin{bmatrix}4 & -2\\-3 & 1\end{bmatrix} = \begin{pmatrix}-1 & \frac{1}{2}\\ \frac{3}{4} & -\frac{1}{4}\end{pmatrix}$$

$$\therefore (KA)^{-1} = \frac{1}{K}A^{-1}$$

(7) $(A^t)^{-1} = (A^{-1})^t$

Let $A = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$, $A^t = \begin{pmatrix}1 & 3\\2 & 4\end{pmatrix}$, $A^{-1} = -\frac{1}{2}\begin{pmatrix}4 & -2\\-3 & 1\end{pmatrix}$

$$(A^t)^{-1} = -\frac{1}{2}\begin{pmatrix}4 & -3\\-2 & 1\end{pmatrix}, \quad (A^{-1})^t = -\frac{1}{2}\begin{pmatrix}4 & -3\\-2 & 1\end{pmatrix}$$

$$\therefore (A^t)^{-1} = (A^{-1})^t$$

(8) $|A^{-1}| = \frac{1}{|A|}$

Let $A = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$, $|A| = -2$, $A^{-1} = -\frac{1}{2}\begin{pmatrix}4 & -2\\-3 & 1\end{pmatrix}$

$$|A^{-1}| = \left|-\frac{1}{2}\begin{pmatrix}4 & -2\\-3 & 1\end{pmatrix}\right| = \begin{vmatrix}-2 & 1\\ \frac{3}{2} & -\frac{1}{2}\end{vmatrix} = 1 - \frac{3}{2} = -\frac{1}{2}$$

$$\frac{1}{|A|} = -\frac{1}{2}$$

$$\therefore |A^{-1}| = \frac{1}{|A|}$$

Example:

Find the value of x and y if $A^{-1} = \begin{pmatrix}-1 & x\\0 & y\end{pmatrix}$, $A = \begin{pmatrix}-1 & 2\\0 & 4\end{pmatrix}$

Solution:

$$A^{-1} = \frac{1}{|A|}adj(A) = \frac{1}{-4}\begin{pmatrix}4 & -2\\0 & -1\end{pmatrix} = \begin{pmatrix}-1 & \frac{1}{2}\\ 0 & \frac{1}{4}\end{pmatrix} = \begin{pmatrix}-1 & x\\0 & y\end{pmatrix}$$

$$\therefore \boldsymbol{x} = \frac{1}{2}, \qquad y = \frac{1}{4}$$

Example:

Let A and B are 3×3 matrices and $|A| = -4, \ \ |B| = 3$. Find

i. $|B^t|$ **ii.** $|2A|$ **iii.** $|A^{-1}|$ **iv.** $|AB|$

Solution:

i. $|B^t| = |B| = 3$

ii. $|2A| = 2^3|A| = (8)(-4) = -32$

iii. $|A^{-1}| = \frac{1}{|A|} = \frac{1}{-4} = -\frac{1}{4}$

iv. $|AB| = |A||B| = (-4)(3) = -12$

Example:

If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$, find

i. $\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$ ii. $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$ iii. $\begin{vmatrix} 2a & 2b & 2c \\ -g & -h & -i \\ 3d & 3e & 3f \end{vmatrix}$

Solution:

i. $\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$

ii. $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -4 \qquad (R_2 \rightleftharpoons R_3)$

iii. $\begin{vmatrix} 2a & 2b & 2c \\ -g & -h & -i \\ 3d & 3e & 3f \end{vmatrix} = (2)(-1)(3)\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = (-6)(-4) = 24$

Tutorial (3)

(1) Find A^{-1} if :

i. $A = \begin{pmatrix} 5 & 3 \\ -1 & 0 \end{pmatrix}$

ii. $A = \begin{pmatrix} -1 & -2 & 3 \\ 2 & 1 & 1 \\ 4 & -1 & 6 \end{pmatrix}$

(2) Find the value of x and y if:

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ x & y \end{pmatrix} \ and\ A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

(3) If $A^{-1} = \begin{pmatrix} 0 & -3 \\ 3 & 1 \end{pmatrix}$ find A.

(4) If $A = \begin{pmatrix} -2 & -1 \\ 3 & -1 \end{pmatrix}$ find A^{-2}.

(5) If $A = \begin{pmatrix} 1 & -6 \\ -1 & 2 \end{pmatrix}$ and B is 2×2 matrix such that $AB = I$ find the matrix B (use the inverse).

(6) If $(3A)^{-1} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$ find A.

(7) If $(A-2I)^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ find A.

SOLUTION OF NONHOMOGENOUS LINEAR EQUATIONS

By the inverse of the matrix:

If A is an invertible $n \times n$ matrix and B is $n \times 1$ matrix then the system of equations $AX = B$ has one solution.

$$\boldsymbol{X = A^{-1}B}$$

Example:

Find the solution of the following equations:

$$x + y + y = 2$$
$$2x + y + z = 1$$
$$x - y + 3z = 0$$

Solution:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$X = A^{-1}B$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix} = (3+1) - (6-1) + (-2-1) = -4 \neq 0$$

$$M_{11} = 4, \qquad M_{12} = 5, \qquad M_{13} = -3$$
$$M_{21} = 4, \qquad M_{22} = 2, \qquad M_{23} = -2$$
$$M_{31} = 0, \qquad M_{32} = -1, \qquad M_{33} = -1$$

$$adj(A) = \begin{pmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{pmatrix}^t = \begin{pmatrix} 4 & -5 & -3 \\ -4 & 2 & 2 \\ 0 & 1 & -1 \end{pmatrix}^t = \begin{pmatrix} 4 & -4 & 0 \\ -5 & 2 & 1 \\ -3 & 2 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj(A) = -\frac{1}{4}\begin{pmatrix} 4 & -4 & 0 \\ -5 & 2 & 1 \\ -3 & 2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} 4 & -4 & 0 \\ -5 & 2 & 1 \\ -3 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} (8-4+0) \\ (-10+2+0) \\ (-6+2+0) \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} 4 \\ -8 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$\therefore (x, y, z) = (-1, 2, 1)$$

The Cramer's Rule:

If A is an invertible $n \times n$ matrix and B is $n \times 1$ matrix and A_j is a matrix obtained A by replacing the j[th] column of A by B then the system of equation $AX = B$ has one solution.

$$x_j = \frac{|A_j|}{|A|}, \quad j = 1, 2, \dots, n \;, \qquad |A| \neq 0$$

Example:

Find the solution of the following equations:

$x + y + y = 2$ $\qquad x_1 + x_2 + x_3 = 2$

$2x + y + z = 1 \quad \Rightarrow \quad 2x_1 + x_2 + x_3 = 1$

$x - y + 3z = 0$ $\qquad x_1 - x_2 + 3x_3 = 0$

Solution:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad x_j = \frac{|A_j|}{|A|}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix} = -4 \neq 0$$

$$|A_1| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 3 \end{vmatrix} = 2(3+1) - (3-0) + (-1-0) = 4$$

$$|A_2| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} = (3-0) - 2(6-1) + (0-1) = -8$$

$$|A_2| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = (0+1) - (0-1) + 2(-2-1) = -4$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{4}{-4} = -1$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{-8}{-4} = 2$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{-4}{-4} = 1$$

Tutorial (4)

Find the solution of the following equations by using:

a) The inverse of a matrix.

b) The Cramer s Rule

i. $x + y + z = 2$

$x - y - z = 0$

$2x + y - z = -1$

ii. $x + y - z = 2$

$3x - 2y = 17$

$7y + 5z = 3.$

iii. $3x + y - z = 2$
$x + 4y + z = 1$
$x + 2y + z = 3.$

iv. $2x - y + z = 8$

$4x + 3y + z = 7$

$6x + 2y + 2z = 15.$