SYSTEMS OF LINEAR EQUATIONS

A linear system of m equations in the variables is a list of m equations of the form:

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \rightarrow (1)$$

Where $x_1, x_2, \dots\dots, x_n$ are variables, $a_{11}, a_{12}, a_{13}, \dots\dots, a_{mn}$ are coefficients and $b_1, b_2, \dots, b_m$ are constants.

The system of the equation (1) is called nonhomogeneous, if $b_1 = b_2 = \cdots = b_m = 0$ called homogeneous system.

Example:

$$\left.\begin{array}{c} 4x + y = 3 \\ x - 5y = 1 \end{array}\right\} \text{Nonhomogeneou} \qquad \left.\begin{array}{c} 4x + y = 0 \\ x - 5y = 0 \end{array}\right\} \text{Homogeneous}$$

The equation (1) can be written in from $AX = B$ where

$A = [a_{ij}]$ The coefficients of variables

$B = \begin{bmatrix} b_1 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}$ The constants $\qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ The variables

The homogeneous system can be written in the from $AX = 0$ where $0 \equiv zero\ matrix.$

THE DETERMINANTS

Every square matrix A has scalar quantity is called the determinants denoted by $|A|$ or $\det A$ and written as.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots\dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots\dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots\dots & a_{mn} \end{vmatrix} \qquad \rightarrow \qquad (1)$$

Definition:

The det of size 2×2 is given by the formula

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Example:

If $A = \begin{pmatrix} 3 & 1 \\ 4 & 5 \end{pmatrix}$ find $|A|$

Solution:

$$|A| = \begin{vmatrix} 3 & 1 \\ 4 & 5 \end{vmatrix} = (3 \times 5) - (4 \times 1) = 15 - 4 = 11$$

Example: find

1) $\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = (-1 \times 1) - (2 \times -3) = -1 + 6 = 5$

2) $\begin{vmatrix} 0 & -3 \\ 4 & -2 \end{vmatrix} = (0 \times -2) - (-3 \times 4) = 0 + 12 = 12$

3) $\begin{vmatrix} 3 & 5 \\ 0 & -1 \end{vmatrix} = (3 \times -1) - (5 \times 0) = -3 - 0 = -3$

4) $\begin{vmatrix} -1 & -2 \\ -5 & -4 \end{vmatrix} = (-1 \times -4) - (-2 \times -5) = 4 - 10 = -6$

5) $\begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix} = (3 \times -4) - (-2 \times 6) = -12 + 12 = 0$

Minors and cofactors:

A new determinant of size $(n-1) \times (n-1)$ obtained from (1).

By deleting the i[th] row and j[th] column is called the minor of the element a_{ij} in the matrix A denoted by M_{ij}.

The cofactors of a_{ij} denoted by A_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$

$$(-1)^{i+j} = \begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots \end{pmatrix}$$

Example:

Find $M_{11}, M_{12}, M_{21}, M_{22}, A_{11}, A_{12}, A_{21}, A_{22}$ if $A = \begin{pmatrix} 5 & -1 \\ -3 & -4 \end{pmatrix}$

Solution:

$$M_{11} = |-4| = -4, \quad M_{12} = |-3| = -3, \quad M_{21} = |-1| = -1, \quad M_{22} = |5| = 5$$
$$A_{11} = +M_{11} = -4, \quad A_{12} = -M_{12} = -(-3) = 3$$
$$A_{21} = -M_{21} = -(-1) = 1, \quad A_{22} = +M_{22} = 5$$

Example:

Find M_{ij}, A_{ij} if $A = \begin{pmatrix} -1 & 2 & 0 \\ -3 & 4 & 1 \\ 5 & 3 & -1 \end{pmatrix}$

Solution:

$$M_{11} = \begin{vmatrix} 4 & 1 \\ 3 & -1 \end{vmatrix} = -4 - 3 = -7, \qquad M_{12} = \begin{vmatrix} -3 & 1 \\ 5 & -1 \end{vmatrix} = 3 - 5 = -2$$

$$M_{13} = \begin{vmatrix} -3 & 4 \\ 5 & 3 \end{vmatrix} = -9 - 20 = -29, \qquad M_{21} = \begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix} = -2 - 0 = -2$$

$$M_{22} = \begin{vmatrix} -1 & 0 \\ 5 & -1 \end{vmatrix} = 1, \qquad M_{23} = \begin{vmatrix} -1 & 2 \\ 5 & 3 \end{vmatrix} = -13, \qquad M_{31} = \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = 2 - 0 = 2$$

$$M_{32} = \begin{vmatrix} -1 & 0 \\ -3 & 1 \end{vmatrix} = -1, \qquad M_{33} = \begin{vmatrix} -1 & 2 \\ -3 & 4 \end{vmatrix} = 2$$

$$A_{11} = +M_{11} = -7, \qquad A_{12} = -M_{12} = 2\,, \quad A_{13} = +M_{12} = -29$$

$$A_{21} = -M_{21} = 2, \qquad A_{22} = +M_{22} = 1\,, \quad A_{23} = -M_{23} = 13$$

$$A_{31} = +M_{31} = 2, \qquad A_{32} = -M_{32} = 1\,, \quad A_{33} = +M_{32} = 2$$

Laplace Expansion:

The determinant of a matrix is equal to the sum of the product obtained by multiplying the elements of the row or column by their respectively cofactors.

$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij}$ (By i^{th} row)

$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij}$ (By j^{th} column)

Example:

Find $|A|$ if:

(1) $A = \begin{pmatrix} 1 & -2 & -3 \\ 2 & 1 & 4 \\ 1 & -1 & 5 \end{pmatrix}$

By $R_1 \Rightarrow |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$$|A| = (1)\begin{vmatrix} 1 & 4 \\ -1 & 5 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$|A| = (5 + 4) + 2(10 - 4) - 3(-2 - 1)$$

$$|A| = 9 + 12 + 9 = 30$$

By $R_2 \Rightarrow |A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$

$$|A| = -(2)\begin{vmatrix} -2 & -3 \\ -1 & 5 \end{vmatrix} + (1)\begin{vmatrix} 1 & -3 \\ 1 & 5 \end{vmatrix} - (4)\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}$$

$$|A| = -2(-10 - 3) + (5 + 3) - 4(-1 + 2)$$

$$|A| = 26 + 8 - 4 = 30$$

By $C_2 \Rightarrow |A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$

$$|A| = -(-2)\begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} + (1)\begin{vmatrix} 1 & -3 \\ 1 & 5 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix}$$

$$|A| = 2(10-4) + (5+3) + (4+6)$$

$$|A| = 12 + 8 + 10 = 30$$

By C_3 $\Rightarrow |A| = (\quad)\begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} - (\quad)\begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} + (\quad)\begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix}$

(2) $A = \begin{pmatrix} 1 & 3 & -5 \\ 0 & -1 & 2 \\ -2 & 0 & 0 \end{pmatrix}$

By R_1 $\Rightarrow$ $|A| = (1)\begin{vmatrix} -1 & 2 \\ 0 & 0 \end{vmatrix} - (3)\begin{vmatrix} 0 & 2 \\ -2 & 0 \end{vmatrix} + (-5)\begin{vmatrix} 0 & -1 \\ -2 & 0 \end{vmatrix}$

$$|A| = (0-0) - 3(0+4) - 5(0-2)$$

$$|A| = 0 - 12 + 10 = -2$$

By R_3 $\Rightarrow$ $|A| = (-2)\begin{vmatrix} 3 & -5 \\ -1 & 2 \end{vmatrix} - (0)\begin{vmatrix} 1 & -5 \\ 0 & 2 \end{vmatrix} + (0)\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix}$

$$|A| = -2(6-5) - 0(2-0) + 0(-1-0)$$

$$|A| = -2 - 0 + 0 = -2$$

By C_2 $\Rightarrow |A| = -(\quad)\begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} + (\quad)\begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} - (\quad)\begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix}$

The properties of the determinants:

(1) If A is square matrix, then $|A^t| = |A|$.

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2, \qquad |A^t| = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 - 6 = -2$$

(2) If two rows (columns) of square matrix are interchanged then the sign of the determinant is changed.

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2, \qquad R_1 \rightleftharpoons R_2 \Rightarrow |A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$$

(3) If a matrix A has zero row (column) then $|A| = 0$.

$$|A| = \begin{vmatrix} 0 & 0 \\ 7 & -1 \end{vmatrix} = 0\ (R_1 = 0), \qquad |A| = \begin{vmatrix} -3 & 0 & 1 \\ 4 & 0 & 9 \\ 5 & 0 & -6 \end{vmatrix} = 0\ (C_2 = 0)$$

(4) If a matrix has two identical rows (columns) then $|A| = 0$.

$$|A| = \begin{vmatrix} 2 & 4 \\ 2 & 4 \end{vmatrix} = 0, \qquad (R_1 = R_2), \qquad |A| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 6 & 2 \\ 3 & -1 & 3 \end{vmatrix} = 0, \qquad (C_1 = C_3)$$

$$|A| = \begin{vmatrix} -3 & 1 \\ 9 & -3 \end{vmatrix} = 0, (C_1 = -3C_3), \qquad |A| = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 6 & -2 \\ 5 & -1 & 4 \end{vmatrix} = 0, \qquad (2R_1 = R_2)$$

(5) If one row (column) of A is multiplied the scalar k to produce A' then $|A'| = k|A|$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, if\ k = 3 \Rightarrow kR_1 = 3R_1 = \begin{pmatrix} 3 & 6 \\ 3 & 4 \end{pmatrix} = A'$$

$$|A'| = \begin{vmatrix} 3 & 6 \\ 3 & 4 \end{vmatrix} = 12 - 18 = -6$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2, \Rightarrow K|A| = 3(-2) = -6$$

(6) $|kA| = k^n|A|$ n: Number of rows or columns.

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$$if\ k = 4 \Rightarrow 4A = \begin{pmatrix} 4 & 8 \\ 12 & 16 \end{pmatrix}$$

$$|4A| = \begin{vmatrix} 4 & 8 \\ 12 & 16 \end{vmatrix} = 64 - 96 = -32$$

$$k^n|A| = 4^2(-2) = -32$$

(7) If A is triangular matrix, then $|A|$ equals the product of its diagonal elements.

$$A = \begin{pmatrix} -1 & 3 & 4 \\ 0 & 2 & -5 \\ 0 & 0 & 3 \end{pmatrix} \Rightarrow |A| = (-1)(2)(3) = -6$$

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 8 & 3 \end{pmatrix} \Rightarrow |A| = (-1)(2)(3) = -6$$

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \Rightarrow |A| = (-1)(2)(3) = -6$$

(8) If I is identity matrix, then $|\mathrm{I}| = 1$.

$$|\mathrm{I}| = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

(9) If A and B are square matrices of the same size then $|AB| = |A||B|$.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow |A| = -2, \qquad B = \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix} \Rightarrow |B| = 5$$

$$\Rightarrow |A||B| = (-2)(5) = -10$$

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -3+4 & -1-2 \\ -9+8 & -3-4 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -1 & -7 \end{pmatrix}$$

$$\Rightarrow |AB| = \begin{vmatrix} 1 & -3 \\ -1 & -7 \end{vmatrix} = -7 - 3 = -10$$

Example:

Use the properties of determinants to find:

1) $\begin{vmatrix} 2 & 5 & 0 \\ 3 & 6 & 0 \\ -1 & 7 & 0 \end{vmatrix} = 0, (\, C_3 = 0)$

2) $\begin{vmatrix} 2 & -3 & 4 \\ 0 & 6 & 0 \\ 3 & 7 & 6 \end{vmatrix} = 0, (2\, C_1 = C_3)$

3) $\begin{vmatrix} 2 & -3 & 4 \\ 0 & 5 & -1 \\ 0 & 0 & 1 \end{vmatrix} = (2)(5)(1) = 10$

Tutorial (2)

(1) Let $A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 0 & 1 \\ -2 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -4 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ Use Laplace expansion to Find:

i. $|A + B|$ **ii.** $|2A|$ **iii.** $|AB|$ **iv.** $|2A^t B|$

(2) Use the properties of the determinants to find.

i. $\begin{vmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 4 & 1 & 5 \end{vmatrix}$

ii. $\begin{vmatrix} 2 & 2 & 3 & 4 \\ -1 & 1 & 4 & 2 \\ 3 & 3 & 1 & 6 \\ 5 & 0 & 2 & 0 \end{vmatrix}$

(3) Prove that (use the properties)

i. $\begin{vmatrix} 1 & 2 & 10 \\ 2 & 3 & 9 \\ 4 & 5 & 11 \end{vmatrix} = -4$

i. $\begin{vmatrix} x & x \\ 4 & 2x \end{vmatrix} = 0$

ii. $\begin{vmatrix} x^2 & 2 & 0 \\ x & 1 & 0 \\ 2 & 1 & -5 \end{vmatrix} = 0$

iii. $\begin{vmatrix} x & 1 & 2 \\ -1 & 0 & 0 \\ 4 & -1 & x-1 \end{vmatrix} = 8$

iv. $\begin{vmatrix} x+3 & 4 \\ 4 & x \end{vmatrix} = \begin{vmatrix} 2 & 4 & 5 \\ 0 & -1 & x \\ x-2 & 0 & 3 \end{vmatrix}$