Linear Algebra and Matrices

Title of Course: Linear Algebra and Matrices
Course Credits: 3 Credits
Course Prerequisites: None
Course Instructor: Dr. Abu baker Ahmed

Topics:

1. Matrices and Matrix Operations. **2 weeks**
2. Determinants. **2 weeks**
3. Systems of Linear Equations. **3 weeks**
4. Vector Spaces. **2 week**
5. Eigenvalues and Eigenvectors. **1 week**
6. Complex numbers. **2 weeks**

MATRICES

Definition:

A rectangle array of the form $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots\dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots\dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots\dots & a_{mn} \end{bmatrix}$, is called a matrix having **m**-rows and **n**-columns denoted by $A, B, C, \dots., A = [a_{ij}]$ where $i = 1,2, \dots m$, $j = 1,2 \dots n.$, a_{ij} are elements of A, and a_{ij} is an element in the ith row and jth column. A matrix with **m**-rows and **n**-columns is called $m \times n$ matrix.

Example:

$$A = \begin{bmatrix} -3 & 4 \\ 1 & -2 \end{bmatrix}_{2\times 2}, \; B = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 5 & -3 \end{bmatrix}_{3\times 2}, \; C = \begin{bmatrix} 7 & -5 & -3 \end{bmatrix}_{1\times 3}, \; D = \begin{bmatrix} 4 \\ -1 \end{bmatrix}_{2\times 1}$$

Equality of Matrices:

Let A and B be two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if these matrices have the same order and the corresponding elements of A and B are equal.

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \; C = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$

$$A = B \;,\quad A \neq C (a_{11} \neq c_{11}) \;,\quad A \neq D \; (the\ order\ is\ different\ ).$$

Example:

Find the value x and y if $A = B$ such that $A = \begin{bmatrix} 1 & -3 \\ x - y & 4 \end{bmatrix}, \; B = \begin{bmatrix} 1 & y \\ 2 & 4 \end{bmatrix}$.

Solution:

$$y = -3 \; \Rightarrow \; x - y = 2 \; \Rightarrow x - (-3) = 2 \; \Rightarrow x = 2 - 3 = -1$$

$$\therefore x = -1 , y = -3$$

Example:

Find the value x, y and z if $\begin{bmatrix} x^2 + 1 \\ 7 \\ 14 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 26 \\ y \\ 3z + 2 \end{bmatrix}_{3\times 1}$.

Solution:

$$x^2 + 1 = 26 \; \Rightarrow x^2 = 25 \; \Rightarrow x = \pm 5$$

$$y = 7 \;,\quad 14 = 3z + 2 \; \Rightarrow 12 = 3z \; \Rightarrow z = 4$$

Addition and Subtraction of Matrices:

Let A and B be two matrices of the same order then the sum of A and B denoted by $(A+B)$ is defined by.

$$A+B=[a_{ij}+b_{ij}]$$

The difference between A and B denoted by $(A-B)$ is defined by.

$$A-B=[a_{ij}-b_{ij}]$$

Laws of Addition and Subtraction of Matrices:

1. $A+B=B+A$

2. $(A+B)+C=A+(B+C)$

3. $A-B=-(B-A)$

Example:

Let $\boldsymbol{A}=\begin{bmatrix}-1 & -3\\ 4 & 5\\ 6 & 0\end{bmatrix}_{3\times2}$, $\boldsymbol{B}=\begin{bmatrix}4 & 0\\ -1 & 2\\ -8 & 5\end{bmatrix}_{3\times2}$, $\boldsymbol{C}=\begin{bmatrix}2 & -1 & 1\end{bmatrix}_{1\times3}$, $\boldsymbol{D}=\begin{bmatrix}-4 & 1 & 2\end{bmatrix}_{1\times3}$

Find: (i) $A+B$ **(ii)** $D-C$ **(iii)** $B+C$

Solution:

(i) $A+B=\begin{bmatrix}-1 & -3\\ 4 & 5\\ 6 & 0\end{bmatrix}_{3\times2}+\begin{bmatrix}4 & 0\\ -1 & 2\\ -8 & 5\end{bmatrix}_{3\times2}=\begin{bmatrix}3 & -3\\ 3 & 7\\ -2 & 5\end{bmatrix}_{3\times2}$

(ii) $D-C=\begin{bmatrix}-4 & 1 & 2\end{bmatrix}_{1\times3}-\begin{bmatrix}2 & -1 & 1\end{bmatrix}_{1\times3}=\begin{bmatrix}-6 & 2 & 1\end{bmatrix}_{1\times3}$

(iii) $B+C=\begin{bmatrix}4 & 0\\ -1 & 2\\ -8 & 5\end{bmatrix}_{3\times2}+\begin{bmatrix}2 & -1 & 1\end{bmatrix}_{1\times3}=$ not found or undefined

Scalar Multiplication:

Let A be $m\times n$ matrix and K is a scalar then the product of the scalar K with the matrix A denoted by KA is defined by the formula.

$$KA=k[a_{ij}]=[ka_{ij}]$$

Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\Rightarrow KA=\begin{bmatrix}ka & kb\\ kc & kd\end{bmatrix}$

Laws of Scalar Multiplication:

Let A and B be matrices and K , L are scalars then.

1. $K(A+B)=KA+KB$

2. $(K+L)A=KA+LA$

3. $(KL)A=K(LA)$

Multiplication of Matrices:

Let A be $m \times p$ matrix and B be $p \times n$ matrix, the multiplicity is defined if the number of columns of A is equal to the number of rows of B. The product of A, B denoted by AB is $m \times n$ matrix, obtained by multiplying the i^{th} row of A by the j^{th} column of B to get matrix C, the elements of $C$are calculated from.

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

Laws of Multiplication:

Let A, B and C are matrices and K is a scalar then.

1. $(AB)C = A(BC)$
2. $A(B+C) = AB + AC$
3. $(B+C)A = BA + CA$
4. $K(AB) = (KA)B = A(KB)$
5. $AB \neq BA$

Example:

Let $\boldsymbol{A} = \begin{bmatrix} 3 & -1 \\ 0 & -1 \\ 2 & 4 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} -1 & 0 & 4 \end{bmatrix}, \boldsymbol{E} = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & 1 \end{bmatrix}$

Find:

(i) AB **(iii)** EC **(v)** DC **(vii)**B^2 **(ix)** C^2
(ii) BA **(iv)** CD **(vi)** BE **(viii)** AE

Solution:

(i) $AB = \begin{bmatrix} 3 & -1 \\ 0 & -1 \\ 2 & 4 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}_{2\times 2} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}_{3\times 2}$

$$= \begin{bmatrix} [(3\times 1) + (-1\times -3)] & [(3\times 2) + (-1\times 1)] \\ [(0\times 1) + (-1\times -3)] & [(0\times 2) + (-1\times 1)] \\ [(2\times 1) + (4\times -3)] & [(2\times 2) + (4\times 1)] \end{bmatrix}$$

$$= \begin{bmatrix} 3+3 & 6-1 \\ 0+3 & 0-1 \\ 2-12 & 4+4 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 3 & -1 \\ -10 & 8 \end{bmatrix}$$

(ii) $BA = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}_{2\times 2} \begin{bmatrix} 3 & -1 \\ 0 & -1 \\ 2 & 4 \end{bmatrix}_{3\times 2}$ not found

(iii) $EC = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} \ \end{bmatrix}_{2\times 1}$

$$= \begin{bmatrix} [(2\times 1) + (-1\times 2) + (0\times 3)] \\ [(3\times 1) + (5\times 2) + (1\times 3)] \end{bmatrix} = \begin{bmatrix} 2-2+0 \\ 3+10+3 \end{bmatrix} = \begin{bmatrix} 0 \\ 16 \end{bmatrix}$$

(iv) $CD = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times\mathbf{1}} \begin{bmatrix} -1 & 0 & 4 \end{bmatrix}_{\mathbf{1}\times 3} = \begin{bmatrix} -1 & 0 & 4 \\ -1 & 0 & 8 \\ -3 & 0 & 12 \end{bmatrix}_{3\times 3}$

(v) $DC = \begin{bmatrix} -1 & 0 & 4 \end{bmatrix}_{1\times\mathbf{3}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathbf{3}\times 1} = [11]_{\mathbf{1\times 1}}$

(vi) $BE = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}_{2\times\mathbf{2}} \begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & 1 \end{bmatrix}_{\mathbf{2}\times 3}$

$$= \begin{bmatrix} (2+6) & (-1+10) & (0+2) \\ (-6+3) & (3+5) & (0+1) \end{bmatrix}_{\mathbf{2}\times 3} = \begin{bmatrix} 8 & 9 & 2 \\ -3 & 8 & 1 \end{bmatrix}$$

(vii) $B^2 = B.B = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}_{2\times\mathbf{2}} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}_{\mathbf{2}\times 2}$

$$= \begin{bmatrix} (1+-6) & (2+2) \\ (-3+-3) & (-6+1) \end{bmatrix}_{\mathbf{2\times 2}} = \begin{bmatrix} -5 & 4 \\ -6 & -5 \end{bmatrix}$$

(viii) $AE = \begin{bmatrix} 3 & -1 \\ 0 & -1 \\ 2 & 4 \end{bmatrix}_{3\times\mathbf{2}} \begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & 1 \end{bmatrix}_{\mathbf{2}\times 3}$

$$= \begin{bmatrix} (6-3) & (-3-5) & (0-1) \\ (0-3) & (0-5) & (0-1) \\ (2+12) & (-2+20) & (0+4) \end{bmatrix}_{3\times 3} = \begin{bmatrix} 3 & -8 & -1 \\ -3 & -5 & -1 \\ 14 & 18 & 4 \end{bmatrix}$$

(ix) $C^2 = C.C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times\mathbf{1}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathbf{3}\times 1}$ not found

The Transpose of a Matrix:

The transpose of a matrix A denoted by A^t is the matrix obtained by writing the rows of A in order as columns. If A is $m \times n$ matrix, then A^t is $n \times m$ matrix.

Laws the Transpose:

1. $(A \pm B)^t = A^t \pm B^t$
2. $(AB)^t = B^t A^t$
3. $(A^t)^t = A$
4. $(KA)^t = KA^t$

Example:

Let $\boldsymbol{A} = \begin{bmatrix} 6 & 4 & -3 \\ 0 & -1 & 2 \end{bmatrix}_{2\times 3}$, $\quad \boldsymbol{B} = \begin{bmatrix} -1 & -3 \\ 2 & -4 \end{bmatrix}_{2\times 2}$, $\quad \boldsymbol{C} = \begin{bmatrix} 5 & 2 & -1 \end{bmatrix}_{1\times 3}$

Find:

(i) A^t **(ii)** B^t **(iii)** C^t

Solution:

(i) $A^t = \begin{bmatrix} 6 & 4 & -3 \\ 0 & -1 & 2 \end{bmatrix}^t_{2\times 3} = \begin{bmatrix} 6 & 0 \\ 4 & -1 \\ -3 & 2 \end{bmatrix}_{3\times 2}$

(ii) $B^t = \begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}_{2\times 2}$

(iii) $C^t = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}_{3\times \mathbf{1}}$

Types of the Matrices:

(1) Zero Matrix:

Is the matrix which its elements are zero, dented by O.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(2) Square Matrix:

For any matrix A if $m = n$ then A is square matrix. A is $n \times n$ matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \quad , \quad \begin{bmatrix} -3 & 4 & 1 \\ 5 & 0 & -2 \\ 6 & -1 & 3 \end{bmatrix}_{3\times 3}$$

The elements a_{ii} in a square matrix are called main diagonal.

$$\begin{bmatrix} a_{11} & & & & \\ & a_{22} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & a_{nn} \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{-3} & 4 & 1 \\ 5 & \mathbf{0} & 2 \\ 2 & -1 & \mathbf{5} \end{bmatrix}$$

(3) Triangular Matrix:

A square matrix that all the elements above (under) the main diagonal are zero is called lower (upper) triangular matrix.

$$\underset{\text{Lower}}{\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \quad , \quad \underset{\text{Upper}}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}}$$

If A and B are triangular matrices the $A \pm B$, AB , BA, are triangular matrices.

(4) Identity Matrix:

A square matrix that all its elements along the diagonal are **1** and another elements are zero is called identity matrix and denoted by **I**.

$$\mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} = \cdots$$

(5) Symmetric Matrix:

A square matrix is called symmetric if $A^t = A$.

If A is asymmetric matrix, then $A^t + A$, AA^t are symmetric

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & -5 \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 3 & \mathbf{2} & -\mathbf{1} \\ \mathbf{2} & 6 & \mathbf{4} \\ -\mathbf{1} & \mathbf{4} & -5 \end{bmatrix}$$

$$a_{12} = a_{21} \text{ , } a_{23} = a_{32} \text{ , } a_{13} = a_{31} \text{ ....}$$

Tutorial (1)

(1) Let $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \\ 5 & -1 \end{bmatrix}$, $\quad B = \begin{bmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{bmatrix}$, $\quad C = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Find:

1. The matrix D such that $A + B - D = 0$. **2.** $B^t C$

(2) Prove that $A^2 = I$ if $A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

(3) Let $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \\ 1 & -1 \end{bmatrix}$, $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\quad C = [1 \quad -3]$, $\quad D = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 3 & 1 \\ 2 & 1 & -3 \end{bmatrix}$

Find:

1. AB **2.** $B^t + C$ **3.** DAB.

(4) Let $A = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -2 & 3 \\ 2 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -3 & 3 \\ 2 & -1 & 2 \\ -3 & 3 & -2 \end{bmatrix}$ Prove that $A^2 = B^2$

(5) Find the value of x and y if $A^2 + 3I = B$ such that $A = \begin{bmatrix} x & 3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 15 \\ 5 & y \end{bmatrix}$

(6) Find the value of x and y if:

i. $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} x & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & y \end{bmatrix}$.

ii. $A = B^t$ Such that $A = \begin{bmatrix} 1 & x \\ 2 & 3 \end{bmatrix}$, $\quad B = \begin{bmatrix} x+y & 2 \\ 2 & 3 \end{bmatrix}$.

(7) Find the value of x, y, z and u if $A = \begin{bmatrix} 1 & y & 2 \\ 3 & 0 & z \end{bmatrix}$, $B = \begin{bmatrix} 1 & x \\ 3 & 0 \\ 1 & u \end{bmatrix}$ such that $(AB)^t =$

$\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$.

(8) Find the matrix B if $ABC = D$ such that

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 8 & 6 & -6 \\ 6 & -1 & 1 \\ -4 & 0 & 0 \end{bmatrix}.$$

(9) Find the matrix B if $AB = I$ and B is 2×2 matrix such that $A = \begin{bmatrix} 1 & -6 \\ -1 & 2 \end{bmatrix}$.

(10) Find the matrix B if $AB = A^t$ such that $A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}$.

(11) Find K if $AB = BA$ such that $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -5 \\ 3 & K \end{bmatrix}$.

(12) Find the value of, y and z if A is symmetric matrix where $A = \begin{bmatrix} 5 & 6 & -3 \\ z & 8 & y \\ x-y & 4 & 1 \end{bmatrix}$

(13) Find the value of a, b and c if $AB = D$ and D is symmetric matrix where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & a \\ c & b \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 5 & 6 \\ 6 & 7 & 0 \end{bmatrix}.$$

(14) Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ Prove that $A^2 - 3A - I = 0$.