جامعة العلوم والتقانة

University of Science and Technology

Faculty of Engineering

Calculus and it’s Applications -1

Lecture - 8

Dr. Husham Abdallah

L'Hopital's Rule

Indeterminant forms $\frac{0}{0}$, $\frac{\infty}{\infty}$

If $f(x)$ and $g(x)$ are two functions and $g(x) \neq 0$ then

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

By using L'Hopital's rule find?

(1) $\lim_{x\to 0}\frac{tanx}{x} = \frac{0}{0}$ $\lim_{x\to 0}\frac{sec^2x}{1} = sec^2 0 = 1$

(2) $\lim_{x\to 0}\frac{1-cosx}{x^2} = \frac{0}{0}$ $\lim_{x\to 0}\frac{sinx}{2x} = \frac{0}{0} = \frac{1}{2}\lim_{x\to 0}\frac{sinx}{x} = \frac{1}{2}$

(3) $\lim_{x\to 2}\frac{x^2-4}{x-2} = \frac{0}{0}$ $\lim_{x\to 2} 2x = 4$

Exercise:

(1) $\lim_{x\to 1}\frac{2x^3-1}{x^2-1}$ (2) $\lim_{x\to\frac{\pi}{2}}\frac{1-sinx}{cosx}$ (3) $\lim_{x\to 0}\frac{x2^x}{2^x-1}$

(4) $\lim\limits_{n\to 0}\frac{x - sinx}{x^3}$.

Implicit Functions

A relationship between x and y is defined as an implicit function, there is no separation of dependent and independent variables.
Written in the form $f(x,y) = 0$, for example $x^2 + y^2 + 4xy + 5 = 0$

Implicit Differentiation

1. Differentiate both side of the equation with respect to x.
2. Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation and move all other terms to the right side of the equation
3. Solve for $\frac{dy}{dx}$

Example: Find $\frac{dy}{dx}$ of the following functions

(1) $x^2 + y^2 = 5$ (2) $x^2 + y^2 = 3xy$

(3) $x^2y - 3y = 5x$ (4) $y^3 - xy^2 + \cos xy = 2$

(5) $y^x = x^y$

Solution

(1) $2x + 2yy^\backslash = 0 \quad \Rightarrow \quad 2y\, y^\backslash = -2x \quad \Rightarrow \quad y^\backslash = \frac{-x}{y}$

(2) $2x + 2yy^\backslash = 3xy^\backslash + 3y \qquad 2yy^\backslash - 3xy^\backslash = 2y - 2x$

$y^\backslash(2y - 3x) = 2y - 2x \quad \Rightarrow \quad y^\backslash = \frac{2y-2x}{2y-3x}$

(3) $x^2y^\backslash + 2xy - 3y^\backslash = 5 \quad y^\backslash(x^2 + 2x) = 5 - 2xy \quad \Rightarrow$

$$y^\backslash = \frac{5-2xy}{x^2+2x}$$

(4) $3y^2\, y^\backslash - x2y\, y^\backslash - y - x\, y^\backslash sinxy - ysinxy = 0$

$y^\backslash(3y^2 - x2y - x\, sinxy) = y + ysinxy$

$$y^\backslash = \frac{y+ysinxy}{3y^2 - x2y - x\, sinxy}$$

(5) $y^x = x^y \qquad xlny = ylnx \qquad x\frac{y^\backslash}{y} + lny = y\frac{1}{x} + y^\backslash lnx$

$$x\frac{y^\backslash}{y} - y^\backslash lnx = y\frac{1}{x} - lny \quad \Rightarrow \quad y^\backslash = \frac{\frac{y}{x} - lny}{\frac{x}{y} - lnx}$$

Exercise:

(1) $e^{xy} - lnxy = x^2$ (2) $x^2y^2 = \frac{y}{x}$ (3) $tanxy = sinxy$

Partial derivative

$$f_x \equiv \frac{\partial f}{\partial x}, \quad f_y \equiv \frac{\partial f}{\partial y}, \quad f_{xx} \equiv \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} \equiv \frac{\partial^2 f}{\partial y^2}, \quad f_{yx} \equiv \left(\frac{\partial \frac{\partial f}{\partial x}}{\partial y}\right), f_{xy} \equiv \left(\frac{\partial \frac{\partial f}{\partial y}}{\partial x}\right)$$

Example: if

1) $f(x,y) = x^2 - y^2 + x + y - xy + 9$

$$f_x \equiv \frac{\partial f}{\partial x} = 2x + 1 - y$$

$$f_{xx} \equiv \frac{\partial^2 f}{\partial x^2} = 2$$

$$f_y \equiv \frac{\partial f}{\partial y} = -2y + 1 - x$$

$$f_{yy} \equiv \frac{\partial^2 f}{\partial y^2} = -2$$

$$f_{yx} \equiv \left(\frac{\partial \frac{\partial f}{\partial x}}{\partial y} \right) = -1$$

$$f_{xy} \equiv \left(\frac{\partial \frac{\partial f}{\partial y}}{\partial x} \right) = -1$$

2) $f(x,y) = lnx - y^2 + \tan(xy) + xsiny + 1$

$$f_x \equiv \frac{\partial f}{\partial x} = \frac{1}{x} - 2y + ysec^2(xy) + siny$$

$$f_{xx} \equiv \frac{\partial^2 f}{\partial x^2} = \frac{-1}{x^2} + y^2 sec^2(xy) tan(xy)$$

$$f_y \equiv \frac{\partial f}{\partial y} = -2y + xsec^2(xy) - xcosy$$

$$f_{yy} \equiv \frac{\partial^2 f}{\partial y^2} = -2 + x^2 sec^2(xy) tan(xy)$$

$$f_{yx} \equiv \left(\frac{\partial \frac{\partial f}{\partial x}}{\partial y} \right) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f_{xy} \equiv \left(\frac{\partial \frac{\partial f}{\partial y}}{\partial x} \right) = \cdots \text{……………………………….}$$

Exercise:

1) $f(x,y) = e^{xy} - y^4 + \sec(xy) + xy + 1$

$$f_x \equiv \frac{\partial f}{\partial x} = \text{……………………………………………………………}$$

$$f_{xx} \equiv \frac{\partial^2 f}{\partial x^2} = \cdots \text{…………………………………………………………….}$$

$$f_y \equiv \frac{\partial f}{\partial y} = \cdots \text{…………………………………………………}$$

$$f_{yy} \equiv \frac{\partial^2 f}{\partial y^2} = \cdots \text{…………………………………………………}$$

$$f_{yx} \equiv \left(\frac{\partial \frac{\partial f}{\partial x}}{\partial y} \right) = \text{…………………………………………}$$

$$f_{xy} \equiv \left(\frac{\partial \frac{\partial f}{\partial y}}{\partial x} \right) = \cdots \text{……………………………….}$$

2) $f(x,y,z) = xyz - x\ln(xy) + \frac{1}{zxy} + 7$

$$f_z \equiv \frac{\partial f}{\partial z} = \text{……………………………………………………………}$$

$$f_{zz} \equiv \frac{\partial^2 f}{\partial z^2} = \cdots \text{…………………………………………………………}$$

$$f_{yzx} = \text{…………………………………………}$$

$$f_{xyz} = \cdots \text{……………………………….}$$