بسم الله الرحمن الرحيم

جامعة العلوم والتقانة

University of Science and Technology

Faculty of Engineering

Calculus and it's Applications -1

Lecture (4)

Indeterminate Forms

Indeterminate forms $\frac{0}{0}, \frac{\infty}{\infty}, 0.\infty, \infty - \infty, 1^{\infty}$ use theorems or mathematical operations, for example factoring, multiplying by the conjugate, dividing numerator and denominator by the highest power of x.

Note: $\frac{A}{0} = \infty$, $\frac{0}{A} = 0$, $\frac{A}{\infty} = 0$, $\frac{\infty}{A} = \infty$, $A \equiv$ any number

Example:

Find

(1) $\lim_{x \to 5} \frac{x^2-25}{x-5}$ (2) $\lim_{x \to 6} \frac{x^2-4x-12}{x-6}$ (3) $\lim_{x \to -3} \frac{x^3+27}{x^2-9}$

(4) $\lim_{x \to \infty} \frac{6x-1}{x^3+5}$ (5) $\lim_{x \to \infty} \frac{x-3}{\sqrt{x^2+4}}$ (6) $\lim_{x \to 7} \frac{\sqrt{x+2}-3}{x-7}$

(7) $\lim_{x \to 0} \frac{(4+x)^2-16}{x}$ (8) $\lim_{x \to 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$

Solution

(1) $\lim_{x \to 5} \frac{x^2-25}{x-5} = \frac{25-25}{5-5} = \frac{0}{0}$

$$\lim_{x \to 5} \frac{x^2-25}{x-5} = \lim_{x \to 5} \frac{(x-5)(x+5)}{(x-5)} = \lim_{x \to 5}(x+5) = 10$$

(2) $\lim_{x \to 6} \frac{x^2-4x-12}{x-6} = \frac{36-24-12}{6-6} = \frac{0}{0}$

$$\lim_{x \to 6} \frac{x^2-4x-12}{x-6} = \lim_{x \to 6} \frac{(x-6)(x+2)}{x-6} = \lim_{x \to 6}(x+2) = 8$$

(3) $\lim_{x \to -3} \frac{x^3+27}{x^2-9} = \frac{-27+27}{9-9} = \frac{0}{0}$

$$\lim_{x \to -3} \frac{x^3+27}{x^2-9} = \lim_{x \to -3} \frac{x^3+3^3}{x^2-3^2} = \lim_{x \to -3} \frac{(x+3)(x^2-3x+9)}{(x-3)(x+3)}$$

$$\lim_{x \to -3} \frac{(x^2-3x+9)}{(x-3)} = \frac{9+9+9}{-3-3} = \frac{27}{-6} = -\frac{9}{2}$$

Note: $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

(4) $\lim_{x\to\infty} \frac{6x-1}{x^3+5} = \frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{\frac{6x}{x^3}-\frac{1}{x^3}}{\frac{x^3}{x^3}+\frac{5}{x^3}} = \lim_{x\to\infty} \frac{\frac{6}{x^2}-\frac{1}{x^3}}{1+\frac{5}{x^3}} = \frac{\frac{6}{\infty}-\frac{1}{\infty}}{1+\frac{5}{\infty}} = \frac{0-0}{1+0} = 0$$

(5) $\lim_{x\to\infty} \frac{x-3}{\sqrt{x^2+4}} = \frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{x-3}{\sqrt{x^2+4}} = \lim_{x\to\infty} \frac{\frac{x}{x}-\frac{3}{x}}{\sqrt{\frac{x^2+4}{x^2}}} = \lim_{x\to\infty} \frac{1-\frac{3}{x}}{\sqrt{1+\frac{4}{x^2}}} = \frac{1-\frac{3}{\infty}}{\sqrt{1+\frac{4}{\infty}}} = \frac{1-0}{\sqrt{1+0}} = 1$$

(6) $\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} = \frac{3-3}{7-7} = \frac{0}{0}$

$$\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} \times \frac{\sqrt{x+2}+3}{\sqrt{x+2}+3} = \lim_{x\to 7} \frac{(x+2)-9}{(x-7)\sqrt{x+2}+3} = \lim_{x\to 7} \frac{1}{\sqrt{x+2}+3} = \frac{1}{6}$$

(7) $\lim_{x\to 0} \frac{(4+x)^2-16}{x} = \frac{16-16}{0} = \frac{0}{0}$

$$\lim_{x\to 0} \frac{(4+x)^2-16}{x} = \lim_{x\to 0} \frac{16+8x+x^2-16}{x} = \lim_{x\to 0} \frac{8x+x^2}{x}$$

$$= \lim_{x\to 0} \frac{x(8+x)}{x} = \lim_{x\to 0} 8 + x = 8 + 0 = 8$$

(8) $\lim_{x\to 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1} = \frac{\frac{1}{2}-\frac{1}{2}}{1-1} = \frac{0}{0}$

$$\lim_{x\to 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1} = \lim_{x\to 1} \frac{\frac{2-(x+1)}{2(x+1)}}{x-1} = \lim_{x\to 1} \frac{1-x}{2(x-1)(x+1)}$$

$$= \lim_{x\to 1} \frac{-(x-1)}{2(x-1)(x+1)} = \lim_{x\to 1} \frac{-1}{2(x+1)} = -\frac{1}{4}$$

Special limits

Cauchy theorem:

$$\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}=\frac{m}{n}a^{m-n}$$

The limit of $\frac{\sin x}{x}$ *as* $x\to 0$

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \quad , \quad \lim_{x\to 0}\frac{x}{\sin x}=1$$

The number *e* :

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e \quad , \quad \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x=e^a \quad , \quad \lim_{x\to 0}(1+ax)^{\frac{1}{x}}=e^a$$

Example:

(1) $\lim_{x\to 5}\frac{x^2-25}{x-5}$ (2) $\lim_{x\to -3}\frac{x^3+27}{x^2-9}$ (3) $\lim_{x\to 0}\frac{3x+\sin x}{x}$

(4) $\lim_{x\to 0}\frac{\sin 5x}{x}$ (5) $\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}$ (6) $\lim_{x\to 0}x\ \cot x$

(7) $\lim_{x\to 0}(1+x)^{\frac{1}{x}+3}$ (8) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{5x}$ (9) $\lim_{x\to 0}(1-2x)^{\frac{3}{x}}$

Solution:

(1) $\lim_{x\to 5}\frac{x^2-25}{x-5}=\frac{25-25}{5-5}=\frac{0}{0}$

$$\lim_{x\to 5}\frac{x^2-25}{x-5}=\lim_{x\to 5}\frac{x^2-5^2}{x-5}=\frac{2}{1}(5)^{2-1}=10$$

(2) $\lim_{x\to -3}\frac{x^3+27}{x^2-9}=\frac{0}{0}$

$$\lim_{x\to -3}\frac{x^3+27}{x^2-9}=\lim_{x\to -3}\frac{x^3-(-3)^3}{x^2-(-3)^2}=\frac{3}{2}(-3)^{3-2}=-\frac{9}{2}$$

(3) $\lim\limits_{x\to 0}\frac{3x+\sin x}{x}=\frac{0+0}{0}=\frac{0}{0}$

$$\lim_{x\to 0}\frac{3x+\sin x}{x}=\lim_{x\to 0}\left(\frac{3x}{x}+\frac{sin\,x}{x}\right)=\lim_{x\to 0}3+\lim_{x\to 0}\frac{\sin x}{x}$$

$$=3+1=4$$

(4) $\lim\limits_{x\to 0}\frac{\sin 5x}{x}=\frac{0}{0}$

$$\lim_{x\to 0}\frac{\sin 5x}{x}\times\frac{5}{5}=5\,\lim_{x\to 0}\frac{sin\,5x}{5x}=5(1)=5$$

(5) $\lim\limits_{x\to 0}\frac{\sin 2x}{\sin 3x}=\frac{\sin 0}{\sin 0}=\frac{0}{0}$

$$\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}\times\frac{x}{x}=\lim_{x\to 0}\frac{sin\,2x}{x}\times\lim_{x\to 0}\frac{x}{sin\,3x}$$

$$=2\lim_{x\to 0}\frac{sin\,2x}{2x}\times\frac{1}{3}\lim_{x\to 0}\frac{3x}{sin\,3x}=2(1)\times\frac{1}{3}(1)=\frac{2}{3}$$

(6) $\lim\limits_{x\to 0}x\ \cot x=0.\cot 0=0.\infty$

$$\lim_{x\to 0}x\ \frac{\cos x}{\sin x}=\lim_{x\to 0}\ \frac{x}{sin\,x}\times\lim_{x\to 0}\cos x=1\times\cos 0=1$$

(7) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}+3}=(1+0)^{\frac{1}{0}+3}=1^{\infty}$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}+3}=\lim_{x\to 0}(1+x)^{\frac{1}{x}}.\lim_{x\to 0}(1+x)^{3}=e(1)=e$$

(8) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{5x}=\left(1+\frac{1}{\infty}\right)^{\infty}=1^{\infty}$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{5x}=\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right)^{5}=e^{5}$$

(9) $\lim\limits_{x\to 0}(1-2x)^{\frac{3}{x}}=(1-0)^{\frac{3}{0}}=1^{\infty}$

$$\lim_{x\to 0}(1-2x)^{\frac{3}{x}}=\left(\lim_{x\to 0}\left(1+(-2x)\right)^{\frac{1}{x}}\right)^{3}=(e^{-2})^{3}=e^{-6}$$