جامعة العلوم والتقانة

University of Science and Technology

Faculty of Engineering

Calculus and it’s Applications -1

Lecture - Four

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The Limit of Functions

Definition:

Suppose that $f(x)$ is defined when x is near the number a then we write

$$\lim_{x \to a} \mathrm{f}(x) \; = \; \mathrm{L}$$

and we say the limit of $f(x)$ as x approaches a equals L

Rule: $\lim_{x \to a} \mathrm{k} \; = \; \mathrm{k} \quad \textit{where } k \quad \textit{is const}$

Properties:

1. $\mathrm{l}\lim_{x \to a} \mathrm{k} \; = \; \mathrm{k} \qquad \mathrm{k}: \; \text{conest}$
2. $\lim_{x \to a} \mathrm{kf}(x) \; = \; \mathrm{k}\lim_{x \to a} \mathrm{f}(x)$
3. $\lim_{x \to a}(\mathrm{f}(x) + \; \mathrm{g}(x)) \; = \lim_{x \to a} \mathrm{f}(x) + \lim_{x \to a} \mathrm{g}(x)$
4. $\lim_{x \to a}(\mathrm{f}(x) . \; \mathrm{g}(x)) \; = \lim_{x \to a} \mathrm{f}(x) . \; \lim_{x \to a} \mathrm{g}(x)$
5. $\lim_{x \to a}(\frac{\mathrm{f}(x)}{g(x)}) \; = \; \frac{\lim_{x \to a} \mathrm{f}(x)}{\lim_{x \to a} \mathrm{g}(x)}$
6. $\lim_{x \to a}(\mathrm{f}(x))^n \; = (\lim_{x \to a} \mathrm{f}(x))^n$
7. $\lim_{x \to a}(\ln \mathrm{f}(x)) = \ln(\lim_{x \to a}(\mathrm{f}(x))$

One Sided Limits

If $f(x)$ approaches L as x approaches a through values less than a $(x < a)$ then we say that L is a limit of $f(x)$ from the left and written

$$\lim_{x \to \bar{a}} \mathrm{f}(x) = L$$

If $f(x)$ approaches L as x approaches a through values greater than a $(x > a)$ then we say that L is a limit of $f(x)$ from the right and written

$$\lim_{x \to a^+} \mathrm{f}(x) = L$$

$f(x) = L$ if and only if $\lim_{x \to a} \mathrm{f}(x) = L$

Rule : we say that the limite fund if $\lim_{x \to \bar{a}} \mathrm{f}(x) = \lim_{x \to a^+} \mathrm{f}(x)$

Example (1):

Find:

(1) $\lim_{x \to 0} x^2 + 4x - 5$ (2) $\lim_{x \to 1} \sqrt{2x + 4}$ (3) $\lim_{x \to 0} \cos x + x^2$

(4) $\lim_{x \to 0} \ln(x + 7)$ (5) $\lim_{x \to 3} \frac{x^2+1}{x-2}$ (6) $\lim_{x \to 0} 9$

Solution

(1) $\lim_{x\to 0} x^2+4x-5 = 0^2+4(0)-5=-5$

(2) $\lim_{x\to 1}\sqrt{2x+4}=\sqrt{2+4}=\sqrt{6}$

(3) $\lim_{x\to 0} \cos x + x^2 = cos0 - 0^2 = 1-0=1$

(4) $\lim_{x\to 0} \ln(x+7) = \ln(0+7) = ln7$

(5) $\lim_{x\to 3}\frac{x^2+1}{x-2}=\frac{9+1}{3-2}=10$

(6) $\lim_{x\to 0} 9 = 9$

Example (2):

If $f(x)=\begin{cases}2x-1, & x\geq 1\\ x^2, & x<2\end{cases}$ Find:

(1) $\lim_{x\to 1^+} \mathrm{f}(x)$ (2) $\lim_{x\to 1} \mathrm{f}(x)$ (3) $\lim_{x\to 1} \mathrm{f}(x)$

Solution

(1) $\lim_{x\to 1^+} \mathrm{f}(x) = \lim_{x\to 1} 2x-1 = 2-1=1$

(2) $\lim_{x\to \bar{1}} \mathrm{f}(x) = \lim_{x\to \bar{1}} \mathrm{x}^2 = 1^2 = 1$

(3) $\lim_{x\to 1} \mathrm{f}(x) = 1$ $\left(\lim_{x\to 1^+} \mathrm{f}(x) = \lim_{x\to \bar{1}} \mathrm{f}(x)\right) = 1$

Exercise – 3

1. Find the domain of definition of the following functions

(1) $f(x)=x^2-3x+9$ (2) $f(x)= cotx - tanx$

(3) $f(x) = \sqrt{4-2x^2}$ (4) $f(x) = \ln(6-5x+x^2)$

(5) $f(x) = \sqrt{|2x - 5|}$ (6) $f(x) = \frac{2x}{x^{2-2x+1}}$

2. Which of the following functions are even and which are odd?

(1) $f(x) = x^6 + 1$ (2) $f(x) = xcotx$ (3) $f(x) = x^2 \sec x$

(4) $f(x) = \frac{tanx}{x^2}$ (5) $f(x) = x^2 - x +$ (6) $f(x) = sinhx$

Indeterminate Forms

Indeterminate forms $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 1^\infty$

theorems or mathematical operations, for example factoring, multiplying by the conjugate, dividing numerator and denominator by the highest power of x.

Note: $\frac{A}{0} = \infty$, $\frac{0}{A} = 0$, $\frac{A}{\infty} = 0$, $\frac{\infty}{A} = \infty$, $A \quad \equiv$ any number

Example:

Rule: $x^2 - a^2 = (x-a)(x+a)$, $(x \pm a)^2 = x^2 \pm 2ax + a^2$,
$(x \pm a)^2 \neq x^2 \pm a^2$, $x^3 \pm a^3 = (x \pm a)(x^2 \mp ax + a^2)$,
$x^2 + a^2 \neq (x-a)(x+a)$

Find

(1) $\lim_{x\to 2}\frac{x^2-4}{x-2}$ (2) $\lim_{x\to 2}\frac{x-2}{2-x}$ (3) $\lim_{x\to 2}\frac{x^2-4x+4}{x-2}$ (4) $\lim_{x\to 3}\frac{x^3+2}{x^2-9}$

(5) lim $\lim_{x\to 1}\frac{x^2-2x+1}{x^2-1}$ (6) $\lim_{x\to 2}\frac{x^2-4}{\sqrt{x+2}-2}$

Solution

(1) $\lim_{x\to 2}\frac{x^2-4}{x-2} = \frac{0}{0} \Rightarrow \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4$

(2) $\lim_{x\to 2}\frac{x-2}{2-x} = \frac{0}{0} \Rightarrow \lim_{x\to 2}\frac{-(2-x)}{2-x} = \lim_{x\to 2} -1 = -1$

(3) $\lim_{x\to 2}\frac{x^2-4x+4}{x-2} = \frac{0}{0} = \Rightarrow \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4$

(4) $\lim_{x\to 3}\frac{x^3+2}{x^2-9} = \frac{29}{0} = \infty$

(5) $\lim_{x\to 1}\frac{x^2-2x+1}{x^2-1} = \frac{0}{0} = \lim_{x\to 1}\frac{(x-1)(x-1)}{(x-1)(x+1)} = \lim_{x\to 1}\frac{x-1}{x+1} = \frac{0}{2} = 0$

$\lim_{x\to 2}\frac{x^2-4}{\sqrt{x+2}-2} = \frac{0}{0} = \lim_{x\to 2}\frac{x^2-4}{\sqrt{x+2}-2}\frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} = \lim_{x\to 2}\frac{(x-2)(x+2)\sqrt{x+2}+2}{x+2-4} =$

$= \lim_{x\to 2}(x+2)\sqrt{x+2} = 16$

Exercise

Find:

(1) $\lim_{x\to\infty}\frac{x^3-x+5}{x^2-2x+6}$ (2) $\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}$ (3) $\lim_{x\to\infty}\frac{4x-3}{x^5-7x+1}$

(4) $\lim_{x\to 1}\frac{(x+1)^2-4}{x-1}$ (5) $\lim_{x\to 0}\frac{x}{\sqrt{x}}$ (6 $\lim_{x\to\infty}\sqrt{x+1}-x$

Special limits:

Cauchy theorem: $\lim_{x\to a}\frac{x^n-a^n}{x-a} = a^{n-1}$ and $\lim_{x\to a}\frac{x^n-a^n}{x^m-a^m} = \frac{n}{m}a^{n-m}$

The limit of:

$\lim_{x\to 0}\frac{sinx}{x} = 1$ *and* $\lim_{x\to 0}\frac{x}{sinx} = 1$ *and* $\lim_{x\to 0}\frac{tanx}{x} = 1$

The number *e* : $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e,\ \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e,$

$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a,\ \lim_{x\to 0}(1+ax)^{\frac{1}{x}} = e^a,$

Example: **finf:**

(1) $\lim_{x\to 3}\frac{x^3-27}{x-3} = \lim_{x\to 3}\frac{x^3-3^3}{x-3} = 3^{3-1} = 8$

(2) $\lim_{x\to 3}\frac{x^3-27}{x^2-9} = \lim_{x\to 3}\frac{x^3-3^3}{x^2-2^2} = \frac{3}{2}3^{3-2} = \frac{9}{2}$

(3) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x} = \lim_{x\to 0}\left(\left(1+\frac{1}{x}\right)^x\right)^3 = e^3$

(3) $\lim_{x\to 0}(1-4x)^{\frac{1}{x}} = e^{-4},$

(4) $\lim_{x\to 0}(1+x)^{\frac{5}{x}} = \left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right)^5 = e^5$

(5)$\lim_{x\to 0}\frac{x^2+sinx}{x} = \lim_{x\to 0}\left(\frac{x^2}{x}+\frac{sinx}{x}\right) = \lim_{x\to 0}x + \lim_{x\to 0}\frac{sinx}{x} = 0+1 = 1$

(6) $\lim_{x\to 0}\frac{sin3x}{x} = \frac{0}{0} \Rightarrow \lim_{3x\to 0}\frac{sinx}{x}\times\frac{3}{3} = 3\lim_{3x\to 0}\frac{sin3x}{3x} = 3\times 1 = 3$

$\lim_{x\to 0}\frac{sin3x}{sin2x} = \frac{0}{0} \Rightarrow \lim_{x\to 0}\frac{sinx}{sin2x}\times\frac{x}{x} =$

(7) $\lim_{x\to 0}\frac{sin3x}{x}\times\frac{x}{x} \Rightarrow 3\lim_{3x\to 0}\frac{sin3x}{3x}\times\frac{1}{2}\lim_{2x\to 0}\frac{2x}{sin2x} = 3\times\frac{1}{2} = \frac{3}{2}$

Exercise

Find the limites:

(1) $\lim_{x\to 0} x\, cotx$ (2) $\lim_{x\to 0} x\, secx$ (3) $\lim_{x\to 0}\frac{sin^2x}{x^2}$ (4) $\lim_{x\to 0}\frac{sinx}{5x}$

(5) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x-4}$ (6) $\lim_{x\to\infty}\left(\frac{2-2x}{1+x}\right)^{3x-4}$

Continuity

A function f(x) is called continuous at a point $x = a$ if satisfies the following conditions

(1) $f(a)$ is defined (2) $\operatorname{llim}_{x\to a} f(x)$ exists ($\operatorname{llim}_{x\to a^+} f(x) = \operatorname{l}\lim_{x\to \bar{a}} f(x)$)

(3) $\lim_{x\to a} f(x) = f(a)$

Example (1):

(1) $f(x) = \frac{1}{x-1}$ $at\; x = 1$

Solution:

$f(1) = \frac{1}{0} = \infty$ not defined

$f(x)$ discontinuous at $x = 1$

(2) $f(x) = f(x) = \begin{cases} 2-x, & x<1 \\ 2x, & x\geq 1 \end{cases}$, at x=1

Solution:

$f(1) = 2$ is define

$\lim_{x\to\bar{1}} 2-x = 2-1 = 1$, $\lim_{x\to 1^+} 2x = 2$

$\lim_{x\to\bar{1}} f(x) \neq \lim_{x\to 1^+} f(x)$ then $f(x)$ is discontinuous at $x = 1$

(3) $f(x) = f(x) = \begin{cases} \frac{sinx}{x}, & x<0 \\ x+1, & x\geq 0 \end{cases}$ $at\; x = 0$

Solution:

$f(0) = 1$ is define

$\lim_{x\to\bar{0}}\frac{sinx}{x} = 1$, $\lim_{x\to 0^+} x+1 = 1$

$\lim_{x\to\bar{1}} f(x) = \lim_{x\to 1^+} f(x)$ then $f(x)$ is continuous at $x = 0$

(4) Find the value of k such that $f(x)$ is continuous at $x = 2$

$$f(x) = \quad f(x) = \begin{cases} kx^2, & x \le 2 \\ 2x + k \quad , & x > 2 \end{cases} \quad at \quad x = 2$$

Solution:

$f(2) = 4k$ is define

$\lim\limits_{x \to 2^-} kx^2 = 4k \qquad , \qquad \lim\limits_{x \to 2^+} 2x + k = 4 + k$

$4k = 4 + k\,, \qquad 3k = 4\,. \quad k = \dfrac{4}{3}$

Exercise:

(1) $f(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3}, & x \ne 3 \\ 5 & , x = 3 \end{cases}$ at $x = 3$

(2) $f(x) = x^2 + 2x - 7 \qquad at\ x = 3$

(3) $f(x) = f(x) = \begin{cases} xtanx, & x < 0 \\ 1 - x^2, & x \ge 0 \end{cases}$, $at \quad x = 0$

Derivative

The derivative of the function $y = f(x)$ with respect to x is the rate of change of y with respect x , and written as $\frac{dy}{dx}$, $\frac{df}{dx}$, $y^\backslash$, $f^\backslash(x)$ and defined by the formula:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

This method is called the first principles.

The process of compute the derivative of a given function is called differentiation

Example:

Find the derivative of the following functions from the first principles (from the definition)

(1) $y = f(x) = 5x + 2$

Solution:

$f(x) = 5x + 2$

$f(x + \Delta x) = 5(x + \Delta x) + 2$

$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{5(x + \Delta x) + 2 - (5x + 2)}{\Delta x} =$

$\lim_{\Delta x\to 0}\frac{5\,x+5\Delta x+2-5x-2}{\Delta x}=\lim_{\Delta x\to 0}\frac{5\Delta x}{\Delta x}=5\lim_{\Delta x\to 0}1=5$

(2) $y = f(x) = k$

Solution:

$f(x) = k$

$f(x + \Delta x) = k$

$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{k-k}{\Delta x} = \lim_{\Delta x\to 0}0=0$

(3) $y = f(x) = x$

Solution:

$f(x) = x$

$f(x + \Delta x) = (x+\Delta)$

$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{(x+\Delta x)-x}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=$

$\lim_{\Delta x\to 0}1=1$

(4) $y = f(x) = x^7$

Solution:

By using Cousy

$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^7-x^7}{\Delta x} =$

$\lim_{x+\Delta x\to x}\frac{(x+\Delta x)^7-x^7}{x+\Delta x-x}=7x^{7-1}=7x^6$

(5) $y=f(x)=sinx$

Solution:

$f(x) = sinx$

$f(x + \Delta x) = sin(x+\Delta)$

$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-sinx}{\Delta x}=$

$\lim_{\Delta x\to 0}\frac{2\cos\frac{(x+\Delta x+x)}{2}-sin\frac{(x+\Delta x-x)}{2}}{\Delta x}=\lim_{\Delta x\to 0}\frac{2\cos\frac{(2x+\Delta x)}{2}sin\frac{(\Delta x)}{2}}{\Delta x}=$

$\lim_{\Delta x\to 0}\cos\frac{(2x+\Delta x)}{2}\frac{\lim\limits_{\Delta x\to 0}sin\frac{(\Delta x)}{2}}{\Delta x}=\cos x\lim_{\Delta x\to 0}\frac{sin\frac{(\Delta x)}{2}}{\frac{\Delta x}{2}}=cosx(1)=$

$cosx$

Exercise:

(1) $y = f(x) = \sqrt{x}$ (2) $y = f(x) = x^2 - 3x + 4$

(3) $y = f(x) = \frac{1}{x-5}$ (4) $y = f(x) = cosx$

(5) $y = f(x) = \sqrt[3]{x+3}$ (6) $y = f(x) = e^x$ (7) $y = f(x) = \log_3 x^2$

Rules of Differentiation

$$\frac{dy}{dx} \cdot\ y^{\setminus},\ \ f^{\setminus}(x)$$

(1) $\frac{dk}{dx} = 0$, k ≡ constant (2) $\frac{dx}{dx} = 1$

(3) $\frac{d(kx)}{dx} = k$ (4) $\frac{dx^n}{dx} = nx^{n-1}$

(5) $\frac{d(kf(x))}{dx} = k\frac{df(x)}{dx} = kf^{\setminus}(x)$

(6) $\frac{d(f(x) \pm g(x))}{dx} = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx} = f^{\setminus}(x) \pm g^{\setminus}(x)$

(7) $\frac{d(f(x).g(x))}{dx} = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx} = f(x).g^{\setminus}(x) \pm g(x)f^{\setminus}(x)$

(8) $\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = \frac{g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}}{(g(x))^2} = \frac{g(x)f^{\setminus}(x) - f(x)g^{\setminus}(x)}{(g(x))^2}$

(9) $\frac{d(f(x))^n}{dx} = n(f(x))^{n-1}\ f^{\setminus}(x)$

Example:

Find $\frac{dy}{dx}$ of the following functions

(1) $y = f(x) = x^2 + 9x - 9$

Solution:

$$y^{\setminus} = 2x + 9$$

(2) $y = f(x) = (x^2 - 4x)^7$

Solution:

$$y^{\setminus} = 7(x^2 - 4x)^6(2x - 4)$$

(3) $y = f(x) = x^2(x^5 - 8)$

Solution:

$$y^{\backslash} = x^2(5x^4) + 2x(x^5 - 8)$$

(4) $y = x^2 sinx$

Solution:

$$y^{\backslash} = x^2 cosx + 2x\ sinx$$

(5) $y = \frac{cosx}{x^2}$

Solution:

$$y^{\backslash} = \frac{x^2(-sinx) - 2x\ cosx}{x^4} = \frac{-x^2 sinx - 2x\ cosx}{x^4}$$

(5) $y = tanx = \frac{sinx}{cosx}$

Solution:

$$y^{\backslash} = \frac{cosx\ \ cosx + sinx\ sinx}{cos^2x} = \frac{cos^2x + sin^2x}{cos^2x} = \frac{1}{cos^2x} = sec^2x$$

$$y = cos(u) \quad \Rightarrow \quad y^{\backslash} = -u^{\backslash} sin(u)$$

$$y = sin(u) \quad \Rightarrow \quad y^{\backslash} = u^{\backslash} cos(u)$$

$$y = tan(u) \quad \Rightarrow \quad y^{\backslash} = u^{\backslash} sec^2(u)$$

$$y = cot(u) \quad \Rightarrow \quad y^{\backslash} = -u^{\backslash} csc^2(u)$$

$$y = sec(u) \quad \Rightarrow \quad y^{\backslash} = u^{\backslash} sec(u)\tan(u)$$

$$y = csc(u) \quad \Rightarrow \quad y^{\backslash} = -u^{\backslash} csc(u)\cot(u)$$

$$y = ln(u) \quad \Rightarrow \quad y^{\backslash} = \frac{u^{\backslash}}{u}$$

$$y = \log_a(u) \quad \Rightarrow \quad y^{\backslash} = \frac{u^{\backslash}}{lna \quad u}$$

$$y = e^u \quad \Rightarrow \quad y^{\backslash} = u^{\backslash}(e^u)$$

$$y = a^u \quad \Rightarrow \quad y^{\backslash} = u^{\backslash}\, lna(\, a^u)$$

Excersise :

Show that if:

$$y = tan(u) \quad \Rightarrow \quad y^{\backslash} = u^{\backslash} sec^2(u)$$

$$y = cot(u) \quad \Rightarrow \quad y^{\backslash} = -u^{\backslash} csc^2(u)$$

$$y = sec(u) \quad \Rightarrow \quad y^{\backslash} = u^{\backslash} sec(u)\tan(u)$$

$$y = csc(u) \quad \Rightarrow \quad y^{\backslash} = -u^{\backslash} csc(u)\cot(u)$$

$$y = e^u \quad \Rightarrow \quad y^{\backslash} = u^{\backslash}(e^u)$$

$$y = a^u \quad \Rightarrow \quad y^{\backslash} = -u^{\backslash}\, lna(\, a^u)$$

Examples:

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