جامعة العلوم والتقانة

University of Science and Technology

Faculty of Engineering

Calculus and it's Applications -1

Lecture - tow

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The Limit of a Functions

Definition:

Suppose that $f(x)$ is defined when x is near the number a then we write

$$\lim_{x\to a} \mathrm{f}(x) = \mathrm{L}$$

and we say the limit of $f(x)$ as x approaches a equals L

Rule: $\lim_{x\to a} \mathrm{k} = \mathrm{k}$ *where k is const*

Properties:

1. $\mathrm{l}\lim_{x\to a} \mathrm{k} = \mathrm{k}$
2. $\lim_{x\to a} \mathrm{kf}(x) = \mathrm{k}\lim_{x\to a} \mathrm{f}(x)$
3. $\lim_{x\to a}(\mathrm{f}(x) + \mathrm{g}(x)) = \lim_{x\to a} \mathrm{f}(x) + \lim_{x\to a} \mathrm{g}(x)$
4. $\lim_{x\to a}(\mathrm{f}(x) \, . \, \mathrm{g}(x)) = \lim_{x\to a} \mathrm{f}(x) \, . \, \lim_{x\to a} \mathrm{g}(x)$
5. $\lim_{x\to a}\left(\frac{\mathrm{f}(x)}{g(x)}\right) = \frac{\lim_{x\to a} \mathrm{f}(x)}{\lim_{x\to a} \mathrm{g}(x)}$
6. $\lim_{x\to a}(\mathrm{f}(x))^n = (\lim_{x\to a} \mathrm{f}(x))^n$
7. $\lim_{x\to a}(\ln \mathrm{f}(x)) = \ln(\lim_{x\to a}(\mathrm{f}(x))$

One Sided Limits

If $f(x)$ approaches L as x approaches a through values less than a $(x < a)$
then we say that L is a limit of $f(x)$ from the left and written

$$\lim_{x\to a^-} \mathrm{f}(x) = L$$

If $f(x)$ approaches L as x approaches a through values greater than a $(x > a)$
then we say that L is a limit of $f(x)$ from the right and written

$$\lim_{x\to a^+} \mathrm{f}(x) = L$$

$f(x) = L$ if and only if $\lim_{x\to a} \mathrm{f}(x) = L$

Rule : we say that the limite fund if $\lim_{x\to a^-} \mathrm{f}(x) = \lim_{x\to a^+} \mathrm{f}(x)$

Example (1):

Find:

(1) $\lim\limits_{x\to 0} x^2+4x-5$ (2) $\lim\limits_{x\to 1}\sqrt{2x+4}$ (3) $\lim\limits_{x\to 0} \cos x + x^2$

(4) $\lim\limits_{x\to 0} \ln(x+7)$ (5) $\lim\limits_{x\to 3} \frac{x^2+1}{x-2}$ (6) $\lim\limits_{x\to 0} 9$

Solution

(1) $\lim\limits_{x\to 0} x^2+4x-5 = 0^2+4(0)-5 = -5$

(2) $\lim\limits_{x\to 1}\sqrt{2x+4} = \sqrt{2+4} = \sqrt{6}$

(3) $\lim\limits_{x\to 0} \cos x + x^2 = \cos 0 - 0^2 = 1-0 = 1$

(4) $\lim\limits_{x\to 0} \ln(x+7) = \ln(0+7) = \ln 7$

(5) $\lim\limits_{x\to 3} \frac{x^2+1}{x-2} = \frac{9+1}{3-2} = 10$

(6) $\lim\limits_{x\to 0} 9 = 9$

Example (2):

If $f(x) = \begin{cases} 2x-1, & x \geq 1 \\ x^2, & x < 2 \end{cases}$ Find:

(1) $\lim\limits_{x\to 1^+} f(x)$ (2) $\lim\limits_{x\to 1} f(x)$ (3) $\lim\limits_{x\to 1} f(x)$

Solution

(1) $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1} 2x-1 = 2-1 = 1$

(2) $\lim\limits_{x\to \bar{1}} f(x) = \lim\limits_{x\to \bar{1}} x^2 = 1^2 = 1$

(3) $\lim\limits_{x\to 1} f(x) = 1$ $\left(\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to \bar{1}} f(x)\right) = 1$

Exercise – 3

1. Find the domain of definition of the following functions

(1) $f(x) = x^2-3x+9$ (2) $f(x) = \cot x - \tan x$

(3) $f(x) = \sqrt{4-2x^2}$ (4) $f(x) = \ln(6-5x+x^2)$

(5) $f(x) = \sqrt{|2x-5|}$ (6) $f(x) = \frac{2x}{x^2-2x+1}$

2. Which of the following functions are even and which are odd

Indeterminate Forms

Indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 1^{∞}

theorems or mathematical operations, for example factoring, multiplying by the

conjugate, dividing numerator and denominator by the highest power of x.

Note: $\frac{A}{0} = \infty$, $\frac{0}{A} = 0$, $\frac{A}{\infty} = 0$, $\frac{\infty}{A} = \infty$, A $\equiv$ any number

Example:

Rule:

$x^2 - a^2 = (x-a)(x+a)$, $(x \pm a)^2 = x^2 \pm 2ax + a^2$,

$(x \pm a)^2 \neq x^2 \pm a^2$, $x^3 \pm a^3 = (x \pm a)(x^2 \mp ax + a^2)$,

$x^2 + a^2 \neq (x-a)(x+a)$

Find

(1) $\lim_{x\to 2} \frac{x^2-4}{x-2}$ (2) $\lim_{x\to 2} \frac{x-2}{2-x}$ (3) $\lim_{x\to 2} \frac{x^2-4x+4}{x-2}$ (4) $\lim_{x\to 3} \frac{x^3+2}{x^2-9}$

(5) lim $\lim_{x\to 1} \frac{x^2-2x+1}{x^2-1}$ (6) $\lim_{x\to 2} \frac{x^2-4}{\sqrt{x+2}-2}$

Solution

(1)

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{0}{0} \quad \Rightarrow \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4$$

(2) $$\lim_{x\to 2}\frac{x-2}{2-x}=\frac{0}{0} \quad \Rightarrow \lim_{x\to 2}\frac{-(2-x)}{2-x}=\lim_{x\to 2}-1=-1$$

(3) $$\lim_{x\to 2}\frac{x^2-4x+4}{x-2}=\frac{0}{0}= \quad \Rightarrow \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}= \lim_{x\to 2}(x+2)=4$$

(4) $$\lim_{x\to 3}\frac{x^3+2}{x^2-9}= \qquad \frac{29}{0}=\infty$$

(5)

$$\lim_{x\to 1}\frac{x^2-2x+1}{x^2-1}=\frac{0}{0}= \lim_{x\to 1}\frac{(x-1)(x-1)}{(x-1)(x+1)}= \lim_{x\to 1}\frac{x-1}{x+1}=\frac{0}{2}=0$$

$$\lim_{x\to 2}\frac{x^2-4}{\sqrt{x+2}-2}= \frac{0}{0}=\lim_{x\to 2}\frac{x^2-4}{\sqrt{x+2}-2}\,\frac{\sqrt{x+2}+2}{\sqrt{x+2}+2}=\lim_{x\to 2}\frac{(x-2)(x+2)\sqrt{x+2}+2}{x+2-4}=$$
$$=\lim_{x\to 2}(x+2)\sqrt{x+2}=16$$

<u>Exercise</u>

Find:

(1) $\lim_{x\to\infty}\frac{x^3-x+5}{x^2-2x+6}$ (2) $\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}$ (3) $\lim_{x\to\infty}\frac{4x-3}{x^5-7x+1}$

(4) $\lim_{x\to 1}\frac{(x+1)^2-4}{x-1}$ (5) $\lim_{x\to 0}\frac{x}{\sqrt{x}}$ (6 $\lim_{x\to\infty}\sqrt{x+1}-x$

Special limits:

Cauchy theorem: $\lim_{x\to a}\frac{x^n-a^n}{x-a} = a^{n-1}$ and

$\lim_{x\to a}\frac{x^n-a^n}{x^m-a^m} = \frac{n}{m}a^{n-m}$

The limit of:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad and \quad \lim_{x\to 0}\frac{x}{\sin x} = 1 \; and \quad \lim_{x\to 0}\frac{\tan x}{x} = 1$$

The number ***e*** :

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e,\; \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e,$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a,\; \lim_{x\to 0}(1+ax)^{\frac{1}{x}} = e^a,$$

Example: finf:

(1) $\lim_{x\to 3}\frac{x^3-27}{x-3} = \lim_{x\to 3}\frac{x^3-3^3}{x-3} = 3^{3-1} = 8$

(2) $\lim_{x\to 3}\frac{x^3-27}{x^2-9} = \lim_{x\to 3}\frac{x^3-3^3}{x^2-2^2} = \frac{3}{2}3^{3-2} = \frac{9}{2}$

(3) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x} = \lim_{x\to 0}\left(\left(1+\frac{1}{x}\right)^x\right)^3 = e^3$

(3) $\lim_{x\to 0}(1-4x)^{\frac{1}{x}} = e^{-4},$

(4) $\lim_{x\to 0}(1+x)^{\frac{5}{x}} = \left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right)^5 = e^5$

(5)$\lim_{x\to 0}\frac{x^2+\sin x}{x} = \lim_{x\to 0}\left(\frac{x^2}{x}+\frac{\sin x}{x}\right) = \lim_{x\to 0}x + \lim_{x\to 0}\frac{\sin x}{x} = 0+1 =$

(6)

$\lim_{x\to 0}\frac{\sin 3x}{x} = \frac{0}{0} \Rightarrow \lim_{3x\to 0}\frac{\sin x}{x}\times\frac{3}{3} = 3\lim_{3x\to 0}\frac{\sin 3x}{3x} = 3\times 1 = 3$

Exercise

Find the limites:

(1) $\lim_{x\to 0} x\cot x$ (2) $\lim_{x\to 0} x\sec x$ (3) $\lim_{x\to 0} \frac{\sin^2 x}{x^2}$

(4) $\lim_{x\to 0} \frac{\sin x}{5x}$ (5) $\lim_{x\to \infty} \left(1+\frac{1}{x}\right)^{3x-4}$

(6) $\lim_{x\to \infty} \left(\frac{2-2x}{1+x}\right)^{3x-4}$