جامعة العلوم والتقانة

University of Science and Technology

Faculty of Engineering

Calculus and it’s Applications -1

Lecture - One

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Numbers System

Natural Numbers: $\mathbb{N} = \{1,2,3, \dots\}$

Integers: $\mathbb{Z} = \{\dots, -3, -2, -1,0,1,2,3, \dots\} = (-\infty, 0, \infty +)$

Rational Numbers: $\mathbb{Q} = \{x : x = \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$

For example $\frac{2}{5}, \frac{-5}{7} . \frac{1}{2}$

Irrational Numbers: $\mathbb{Q}^*$: numbers which cannot be expressed as the quotient of two integers ($\frac{a}{b}$). $\mathbb{Q} = \{x : x \neq \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$

For example $\sqrt{2}$, $\sqrt[3]{7}$

Real Numbers: $\mathbb{R}$**:**

The set of numbers which are either rational or irrational is called the set of real numbers

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^*$$
$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

For any real numbers a and b we must have $> b , a = b , a < b$

The real numbers can be represented as points on a line

$\mathbb{R}-$ **0** $\mathbb{R}+$

Negative real numbers 0 positive real numbers

Nonnegative real numbers $\mathbb{R}+ \cup \{0\}$, $x \geq 0$

Nonpositive real numbers $\mathbb{R}- \cup \{0\}$, $x \leq 0$

The absolute Value (Modulus):

The absolute value of a real numbers x written $|x|$ is a non-negative real number that satisfies the conditions

$$|x| = x \quad if \quad x \geq 0$$

$$|x| = -x \quad if \quad x < 0$$

For example, $|\pm 3| = 3\,,\quad |0| = 0\,,\quad |-3| = -(-3) = 3$

Properties:

(1) $|a \pm b| = |b \pm a|$
(2) $|ab| = |a||b|$
(3) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}\,, b \neq 0$
(4) $|a + b| \leq |a| + |b|$
(5) $|a - b| \geq |a| - |b|$
(6) $If \quad |x| = a \quad then \quad x = \pm a$
(7) $If \quad |x| \geq a \quad then \quad x \geq a \quad or \quad x \leq -a$
(8) $If \quad |x| \leq a \quad then \quad -a \leq x \leq a$

(9) $|x| = \begin{cases} x \quad , & x \geq 0 \\ -x \quad , & x < 0 \end{cases}$

(10) $|x - a| = \begin{cases} x - a \quad , & x \geq a \\ a - x \quad , & x < a \end{cases}$

(11). $-|a| \leq a \leq |a|$

Example:

Find the value of x if $|2x - 1| = 5$

Solution

$$|2x - 1| = 5 \quad \text{then} \quad 2x - 1 = \pm 5 \quad \text{(from 6)}$$

$$2x - 1 = 5 \quad \rightarrow \quad 2x = 6 \rightarrow x = 3$$

$$2x - 1 = -5 \quad \rightarrow \quad 2x = -4 \rightarrow x = -2$$

Intervals and Inequalities

Closed Interval:

A set of numbers x satisfying the inequality $a \leq x \leq b$ is called a closed interval and denoted by $[a, b]$

$$[\boldsymbol{a}, \boldsymbol{b}] = \{x \in \mathbb{R}: a \leq x \leq b\}$$

$$[1,5] = \{1,23.4.5\}$$

Open Interval:

A set of numbers x satisfying the inequality $a < x < b$ is called an open interval and denoted by (a, b)

$$(\boldsymbol{a}, \boldsymbol{b}) = \{x \in \mathbb{R}: a < x < b\}$$

$$(1,5) = \{2,3,4\}$$

Half Interval:

A set of numbers x satisfying the inequality $a \leq x < b$ or $a < x \leq b$ is called a half-closed interval or half open interval and denoted by $[a, b)$ or $(a, b]$

$$[\boldsymbol{a}, \boldsymbol{b}) = \{x \in \mathbb{R}: a \leq x < b\}$$

$$[1,5) = \{1,2,3,4\}$$

$$(\boldsymbol{a}, \boldsymbol{b}] = \{x \in \mathbb{R}: a < x \leq b\}$$

$$(1,5] = \{2,3,4,5\}$$

Open Infinite Intervals:

$$(\boldsymbol{a}, \infty) = \{x \in \mathbb{R}: x > a\}$$

$$(-\infty, \boldsymbol{b}) = \{x \in \mathbb{R}: x < b\}$$

Infinite Interval or Real Numbers:

$$(-\infty, \infty) = \{x \in \mathbb{R}: -\infty < x < \infty\} \equiv \mathcal{R}$$

Example:

1- Find the set of solution of the following inequality:

$$-13 \leq 3x + 2 \leq 20$$

Solution

$$-13 - 2 \leq 3x \leq 20 - 2$$

$$-15 \leq 3x \leq 18$$

$$-5 \leq x \leq 6$$

The set of solution is $[-5,6]$

2- Find the set of solution of the following inequality

$|x-5| \le 6$ then $-6 \le x-5 \le 6 \rightarrow -1 \le x \le 11 \rightarrow [-1,11]$

The set of solution is $[-1,11]$

Exercise-1 :

Solve the following inequalities:

1. $2x - 3 \ge 9$
2. $|2x - 7| \le 3$
3. $|1 - 2x| < 5$

Function

Definition:

A function is relationship between two variables x and y and written $y = (x)$ where the variable x is called independent variable and the variable y is called dependent variable.

Example:

If $(x) = 3x^3 - 4x^2 - 3x - 7$, $g(x) = \frac{4-5x}{x+2}$ find:

(1) $f(a)$ (2) $f(\frac{1}{2})$ (3) $g(-1)$ (4) $f(0) + g(2)$

Solution

(1) $(a) = 3a\ 3 - 4a\ 2 - 3a - 7$

(2) $f\,(\frac{1}{2}) = 3(\frac{1}{2})^3 - 4(\frac{1}{2})^2 - 3\,(\frac{1}{2}) - 7 = -\frac{73}{8}$

(3) $g(-1) = \frac{4-5(-1)}{(-1)+2} = 9$

(4) $f(0) + g(2) = (3(0) - 4(0) - 3(0) - 7) + \frac{4-5(2)}{2+2} =$

$$-7 + \left(-\frac{6}{4}\right) = -\frac{17}{2}$$

Types of Functions

1. Polynomial Functions :

$$f(x) = a_0 + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$$

Is called a polynomial function of degree n, where $a_0, a_1, a_2, \dots, a_n$ are constants for example, $f(x) = 4x^3 - x^2 + 4x + 2$ polynomial of degree 2 . If $n = 0$

$f(x)$ is a constant function for example, $f(x) = k, \quad where\ k\ is\ const$

2. Rational Functions:

Are the functions in the form $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where $p(x)\ and\ q(x)$ are polynomial functions For example, $f(x) = \frac{x^2+x-2}{x-3}$

3. Exponential Functions

$(i)\ f(x) = a^x$, $a \equiv bas$, $(a > 0$, $a \neq 1)$ For example, $f(x) = 4^x$

$(ii)\ f(x) = e^x$, $e \approx 2.7182\ldots$ where e is called the natural base

Properties:

(1) $a^x a^y = a^{x+y}$ (2) $\frac{a^x}{a^y} = a^{x-y}$ (3) $(a^x)^y = a^{xy}$

(4) $a^{-x} = \frac{1}{a^x}$ (5) $a^x b^x = (ab)^x$ (6) $\frac{a^x}{b^x} = (\frac{a}{b})^x$

Example:

If $f(x) = 2^x$ prove that $f(x+3) - f(x-1) = \frac{15}{2} f(x)$

Solution

$$f(x+3) = 2^{x+3} = 2^x . 2^3 = 8(2^x)$$

$$f(x-1) = 2^{x-1} = 2^x\, 2^{-1}$$

$$f(x+3) - f(x-1) = 8(2^x) - 2^{-1}(2^x) = \frac{15}{2}(2^x)$$

$$f(x+3) - f(x-1) = \frac{15}{2} f(x)$$

4. Logarithmic Functions:

$(i)\ f(x) = log_a\, x$, $\quad a \equiv base \qquad (ii)\ f(x) = log_e\, x = ln\, x$

ln x is called the natural logarithm For example, $f(x) = log_e x$

Properties:

(1) $log(xy) = log\, x + log\, y$ (2) $log\,(\frac{x}{y}) = log\, x - log\, y$

(3) $log\, x^n = n\, log\, x$ (4) $log\,(\frac{1}{x}) = log\,(x)^{-1} = -\,log\, x$

(5) $log_a 1 = 0$ (6) $log_a a = 1$

(7) $ln\, e = 1$ (8) $ln\, e^x = x$ (9) $e^{ln\, x} = x \quad , x > 0$

5. Trigonometric Functions:

$f(x) = sin\, x$, $f(x) = cos\, x$, $f(x) = tan\, x$

$f(x) = cot\, x$, $f(x) = sec\, x$, $f(x) = csc\, x$ or $cosecx$

Laws:

(1) $sin^2 x + cos^2 x = 1$

(2) $1 + tan^2 x = sec^2 x$

(3) $sin\, 2x = 2\, sin\, x\, cos\, x$

(4) $cos\, 2x = 1 - 2\, sin^2 x = 2cos^2 x - 1 = cos^2 x - sin^2 x$

(5) $tan\, 2x = \frac{2tan\, x}{1-tan^2 x}$

(6) $sin(x \pm y) = sin\, x\, cos\, y \pm cos\, x\, sin\, y$

(7) $cos(x \pm y) = cos\, x\, cos\, y \mp sin\, x\, sin\, y$

(8) $sin\, x + sin\, y = 2\, sin\, \frac{x+y}{2}\, cos\, \frac{x-y}{2}$

(9) $sin\, x - sin\, y = 2\, cos\, \frac{x+y}{2}\, sin\, \frac{x-y}{2}$

(10) $cos\, x + cos\, y = 2\, cos\, \frac{x+y}{2}\, cos\, \frac{x-y}{2}$

(11) $cos\, x - cos\, y = 2\, sin \frac{x+y}{2}\, sin\, \frac{y-x}{2}$

6. Inverse Trigonometric Functions

$f(x) = \sin^{-1} x$, $f(x) = \cos^{-1} x$, $f(x) = \tan^{-1} x$

$f(x) = \cot^{-1} x$, $f(x) = \sec^{-1} x$, $f(x) = \csc^{-1} x$

If $y = \sin^{-1} x$ then $x = sin\, y$

$sin(\sin^{-1} x) = x$, $\sin^{-1} (sin\, x) = x$

7. Hyperbolic Functions:

$f(x) = sinh\, x$, $f(x) = cosh\, x$, $f(x) = tanh\, x$

$$f(x) = \coth x\,,\ f(x) = \operatorname{sech} x\,,\ f(x) = \operatorname{csch} x$$

$$\sinh x = \frac{e^x - e^{-x}}{2}\ ,\ \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}\ ,\ \coth x = \frac{\cosh x}{\sinh x} =- \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}\quad ,\quad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x + e^{-x}}$$

Laws:

(1) $\cosh^2 x - \sinh^2 x = 1$ (2) $\operatorname{sech}^2 x = 1 - \tanh^2 x$

(3) $\operatorname{csch}^2 x = \coth^2 x - 1$

8. Inverse Hyperbolic Functions

$$f(x) = \sinh^{-1} x\,,\qquad f(x) = \cosh^{-1} x\ ,\qquad f(x) = \tanh^{-1} x$$

$$f(x) = \coth^{-1} x\qquad ,f(x) = \operatorname{sech}^{-1} x\ ,\qquad f(x) = \operatorname{csch}^{-1} x$$

If $y = \sinh^{-1} x$ then $x = \sinh y$

$$\sinh(\sinh^{-1} x) = x\,, \sinh^{-1}(\sinh x) = x$$

9. Algebraic Functions

$$p_0(x)y^n + p_1(x)y^{n-1} + p_2(x)y^{n-2} + \cdots + p_n(x) = 0$$

is called algebraic Function, where $p_0(x), p_1(x), p_2(x), \dots, p_n(x)$ are polynomial functions

for example, $x^4y^3 + 2x^3y - 3x = 0$

Exercise -2-

1. If $f(x) = 5^x$ Find:

$(i) f(2)$ $\qquad (ii) f(-2)$ $\qquad (iii)\ f\left(\frac{1}{2}\right)$ $\qquad (iv)\ \left(\frac{f(x+1)}{f(x-2)}\right)$

2. Prove that $e^{\ln x} = x$

3. Prove that $\sinh x + \cosh x = e^x$?

Domain of Definition

The set of values of x for which the values of y are determined by the rule $f(x)$ is called the domain of definition of the function, denoted by $D_{f\in(-\infty,+\infty)}$.

(1) If $f(x)$ is polynomial function then $D_f = \mathbb{R}$

(2) If $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ then $D_f = \mathbb{R} - \{q(x) = 0\}$

(3) If $f(x) = \sqrt{p(x)}$ then $D_f = p(x) \geq 0$

(4) If $f(x) = \frac{p(x)}{\sqrt{q(x)}}$ then $D_f = q(x) > 0$

(5) If $f(x) = ln(p(x))$ then $D_f = p(x) > 0$

(6) If $f(x)$ is Exponential function then $D_f = \mathbb{R}$

Example:

Find the domain of definition of the following functions

(1) $f(x) = \frac{x-4}{x-3}$ (2) $f(x) = x^4 - 7x + 2$ (3) $f(x) = \sqrt{x^2 - 4}$

(4) $f(x) = \frac{x-2}{\sqrt{2x-6}}$ (5) $f(x) = \frac{1}{x^2-2x}$ (6) $f(x) = ln(x + 5)$

Solution

(1) $f(x) = \frac{x-4}{x-3} \Rightarrow x - 3 \neq 0 \Rightarrow x \neq 3 \Rightarrow D_f = \{x \in \mathbb{R}: x \neq 3\}$

(2) $f(x) = x^4 - 7x + 5 \Rightarrow D_f = \mathbb{R}$ (polynomial)

(3) $f(x) = \sqrt{x^2 - 4} \Rightarrow x^2 - 4 \geq 0 \Rightarrow x^2 \geq 4 \Rightarrow$

$Df = \{x \in \mathbb{R}: x \geq \pm 2\}$

(4) $f(x) = \frac{x-2}{\sqrt{2x-6}} \Rightarrow 2x - 6 > 0 \Rightarrow 2x > 6 \Rightarrow D_f = \{x \in \mathbb{R}: x > 3\}$

(5) $f(x) = \frac{1}{x^2-2x} \Rightarrow x^2 - 2x \neq 0 \Rightarrow x(x-2) \neq 0 \Rightarrow$

$D_f = \{x \in \mathbb{R}: x \neq \{0,2\}\}$

(6) $f(x) = ln(x + 5) \Rightarrow x + 5 > 0 \Rightarrow x > -5 \Rightarrow$

$D_f = \{x \in \mathbb{R}: x > -5\}$

Even and Odd Functions

A function $f(x)$ is said to be an even if $f(-x) = f(x)$ and is said to be an odd if $f(-x) = -f(x)$

Example:

Which of the following functions are even and which are odd?

(1) $f(x) = x^3 + 2x$ (2) $f(x) = x^4 + 2x^2$

(3) $f(x) = \sin x$ (4) $f(x) = x\,tanx$ (5) $f(x) = x^2 - 3x + 4$

Solution

(1) $f(x) = x^3 + 2x$

$f(-x) = (-x)^3 + 2(-x) - 3 = -x^3 - 2x = -(x^3 + 2x)$

$= -f(x)$

$\therefore -f(x)$ is an odd function

(2) $f(x) = x^4 + 2x^2$

$f(-x) = (-x)^4 + 2(-x)^2 = x^4 + 2x^2 = x^4 + 2x^2 = f(x)$

$\therefore f(x)$ is an even function

(3) $f(x) = \sin x$

$f(-x) = sin(-x) = sin(0 - x)$

$= sin\,0\,cos\,x - sin\,x\,cos\,0 = 0 - sin\,x$

$= -sin\,x = -f(x)$

$\therefore f(x)$ is an odd function

(4) $f(x) = xtanx$

$$f(-x) = xtanx = \frac{x\,sinx}{cosx} = \frac{(-x)\sin(-x)}{\cos(-x)} = \frac{x\,sinx}{cosx} = f(x)$$

$\therefore f(x)$ is not odd and not even

Rule: $\boldsymbol{\sin(-x) = -sinx}$, $\boldsymbol{\cos(-x) = cosx}$

(5) $f(x) = x^2 - 3x + 4$

$f(-x) = (-x)^2 + 3(-x) + 4 = x^2 - 3x + 4 \neq f(x)$

$\therefore f(x)$ is not odd and not even function

The Limit of a Functions

Definition:

Suppose that $f(x)$ is defined when x is near the number a then we write

$\lim_{x\to a} \mathrm{f}(x) = \mathrm{L}$

and we say the limit of $f(x)$ as x approaches a equals L

Rule: $\lim_{x\to a} \mathrm{k} = \mathrm{k}$ $where\ k$ $is\ const$

Properties:

1. l$\lim_{x\to a} \mathrm{k} = \mathrm{k}$

2. $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x)$

3. $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

4. $\lim_{x\to a}(f(x) . g(x)) = \lim_{x\to a} f(x) . \lim_{x\to a} g(x)$

5. $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

6. $\lim_{x\to a}(f(x))^n = (\lim_{x\to a} f(x))^n$

7. $\lim_{x\to a}(\ln f(x)) = \ln(\lim_{x\to a}(f(x))$

One Sided Limits

If $f(x)$ approaches L as x approaches a through values less than a ($x < a$) then we say that L is a limit of $f(x)$ from the left and written

$\lim_{x\to a^-} f(x) = L$

If $f(x)$ approaches L as x approaches a through values greater than a ($x > a$) then we say that L is a limit of $f(x)$ from the right and written

$\lim_{x\to a^+} f(x) = L$

$f(x) = L$ if and only if $\lim_{x\to a} f(x) = L$

Rule : we say that the limite fund if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

Example (1):

Find:

(1) $\lim_{x\to 0} x^2 + 4x - 5$ (2) $\lim_{x\to 1}\sqrt{2x+4}$ (3) $\lim_{x\to 0} \cos x + x^2$

(4) $\lim_{x\to 0} \ln(x+7)$ (5) $\lim_{x\to 3}\frac{x^2+1}{x-2}$ (6) $\lim_{x\to 0} 9$

Solution

(1) $\lim_{x\to 0} x^2 + 4x - 5 = 0^2 + 4(0) - 5 = -5$

(2) $\lim_{x\to 1}\sqrt{2x+4} = \sqrt{2+4} = \sqrt{6}$

(3) $\lim_{x\to 0} \cos x + x^2 = \cos 0 - 0^2 = 1 - 0 = 1$

(4) $\lim_{x\to 0} \ln(x+7) = \ln(0+7) = \ln 7$

(5) $\lim_{x\to 3}\frac{x^2+1}{x-2} = \frac{9+1}{3-2} = 10$

(6) $\lim\limits_{x\to 0} 9 = 9$

Example (2):

If $f(x) = \begin{cases} 2x-1, & x \geq 1 \\ x^2, & x < 2 \end{cases}$ Find:

(1) $\lim\limits_{x\to 1^+} \mathrm{f}(x)$ (2) $\lim\limits_{x\to 1} \mathrm{f}(x)$ (3) $\lim\limits_{x\to 1} \mathrm{f}(x)$

Solution

(1) $\lim\limits_{x\to 1^+} \mathrm{f}(x) = \lim\limits_{x\to 1} 2x - 1 = 2 - 1 = 1$

(2) $\lim\limits_{x\to \bar{1}} \mathrm{f}(x) = \lim\limits_{x\to \bar{1}} \mathrm{x}^2 = 1^2 = 1$

(3) $\lim\limits_{x\to 1} \mathrm{f}(x) = 1$ $\left(\lim\limits_{x\to 1^+} \mathrm{f}(x) = \lim\limits_{x\to \bar{1}} \mathrm{f}(x)\right) = 1$

Exercise – 3

1. Find the domain of definition of the following functions

(1) $f(x) = x^2 - 3x + 9$ (2) $f(x) = cotx - tanx$

(3) $f(x) = \sqrt{4 - 2x^2}$ (4) $f(x) = \ln(6 - 5x + x^2)$

(5) $f(x) = \sqrt{|2x - 5|}$ (6) $f(x) = \frac{2x}{x^{2-2x+1}}$

2. Which of the following functions are even and which are odd?

(1) $f(x) = x^6 + 1$ (2) $f(x) = xcotx$ (3) $f(x) = x^2 \sec x$

(4) $f(x) = \frac{tanx}{x^2}$ (5) $f(x) = x^2 - x +$ (6) $f(x) = sinhx$

Indeterminate Forms

Indeterminate forms

theorems or mathematical operations, for example factoring, multiplying by the

conjugate, dividing numerator and denominator by the highest power of x.

Note:

$\equiv$ any number

Example: