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INTEGRATION التكــــــــــــــــامل

(5) INVERSE TRIGONOMETRIC FUNCTIONS: الدوال المثلثية العكسية

التكاملات الاتية دوال مثلثية عكسية وجميع النتائج صحيحة عند $a \neq 0$

Integrals evaluated with inverse trigonometric functions the following formula hold for any constant $a \neq 0$

(1) $\int \frac{1}{\sqrt{a^2-x^2}}dx = sin^{-1}\frac{x}{a} = cos^{-1}\frac{x}{a}$ $\quad (valid \quad x^2 < a^2)$

(2) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}tan^{-1}\frac{x}{a} =-\frac{1}{a}cot^{-1}\frac{x}{a}$ $\quad (valid\ for\ all\ x)$

(3) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}sec^{-1}\frac{x}{a} =-\frac{1}{a}csc^{-1}\frac{x}{a}$ $\quad (valid\ |x| > a > 0)$

Examples:

Evaluate the following integrals: احسب التكاملات الاتية :

(1) $\int \frac{dx}{x^2+16}$ (2) $\int \frac{dx}{\sqrt{9-x^2}}$ (3) $\int \frac{dx}{x\sqrt{x^2-4}}$ (4) $\int (sin3x + \frac{6}{x^2+5})\,dx$

(5) $\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}}$ (6) $\int_0^1 \frac{dx}{1+x^2}$ (7) $\int \frac{dx}{\sqrt{3-4x^2}}$ (8) $\int_{\frac{2}{\sqrt{3}}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}}$

Solutions:

(1) $\int \frac{dx}{x^2+16} = \frac{1}{4}\tan^{-1}(\frac{x}{4}) + c$

(2) $\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}(\frac{x}{3}) + c$

(3) $\int \frac{dx}{x\sqrt{x^2-4}} =-\frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right) + c$

(4) $\int (sin3x + \frac{6}{x^2+5})\,dx =-\frac{1}{3}cos3x + \frac{6}{\sqrt{5}}\tan^{-1}(\frac{x}{\sqrt{5}}) + c$

(5) $\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}} = [\sin^{-1} x]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

(6) $\int_0^1 \frac{dx}{1+x^2} = (\tan^{-1} x)_0^1 =- \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

(7) $\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{(\sqrt{3})^2-u^2}} = \frac{1}{2}\left[\sin^{-1}\left(\frac{u}{\sqrt{3}}\right)\right] = \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + c$ Where

$a = \sqrt{3}$ and $u = 2x \to du = 2dx \to dx = \frac{du}{2}$

(8) $\int_{\frac{2}{\sqrt{3}}}^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} = [\sec^{-1} x]_{\frac{2}{\sqrt{3}}}^{\sqrt{2}} = \left(\frac{\pi}{4}\right) - \left(\frac{\pi}{6}\right) = \frac{\pi}{12}$

Example:

By completing the square evaluate باكمال المربع احسب:

(1) $\int \frac{dx}{\sqrt{4x-x^2}}$ (2) $\int \frac{dx}{x^2+4x+20}$ (3) $\int \frac{dx}{\sqrt{e^{2x}-6}}$

Solution:

(1) $\int \frac{dx}{\sqrt{4x-x^2}}$

The expression $\sqrt{4x-x^2}$ does not match any of the formula of integration so we first rewrite $4x-x^2$ by completing the square: $4x - x^2 =- (x^2-4x) =- (x^2-4x+4) + 4 =- [(x-2)^2] + 4 = 4-(x-2)^2$

$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}}$ then we substitute $a = 2$ and $u = (x-2)$ $du = dx$

$\therefore \int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}} = \int \frac{du}{\sqrt{4-u^2}} = \tan^{-1}\left(\frac{u}{2}\right) + c = \sin^{-1}\left(\frac{x-2}{2}\right) + c$

(2) $\int \frac{dx}{x^2+4x+20}$ we complete the square on the $x^2+4x+20 = ((x+2)^2-4+20)$

$= ((x+2)^2 + 16)$

$\therefore \int \frac{dx}{x^2+4x+20} = \int \frac{du}{u^2+16} = \frac{1}{4}\tan^{-1}\left(\frac{u}{4}\right) + c = \frac{1}{4}\tan^{-1}\left(\frac{x-2}{4}\right) + c$ let $u = x+2 \to du = dx$

(3) $\int \frac{dx}{\sqrt{e^{2x}-6}} = \int \frac{\frac{du}{e^x}}{\sqrt{u^2-6}} = \int \frac{du}{e^x\sqrt{u^2-6}} = \int \frac{du}{u\sqrt{u^2-6}} = \frac{1}{\sqrt{6}}\sec^{-1}\left(\frac{u}{\sqrt{6}}\right) = \frac{1}{\sqrt{6}}\sec^{-1}\left(\frac{e^x}{\sqrt{6}}\right) + c$

By using substitution $u = e^x \to ud = e^x dx \to dx = \frac{du}{e^x}$

Exercise:**5**

Evaluate the following integrals   احسب التكاملات الاتية:

(1) $\int \frac{dx}{\sqrt{9-x^2}}$ (2) $\int \frac{dx}{\sqrt{1-4x}}$ (3) $\int \frac{dx}{17+x^2}$ (4) $\int \frac{dx}{x\sqrt{25x^2-2}}$

(5) $\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}}$ (6) $\int \frac{dx}{2+(x-1)^2}$ (7) $\int \frac{csc^2 x dx}{1+(cotx)^2}$ (8) $\int \frac{2cosxdx}{1+(sinx)^2}$

(9) $\int \frac{4dx}{t(1+ln^2 x)}$ (10) $\int \frac{sec^2 xdx}{\sqrt{1-tan^2 x}}$ (11) $\int \frac{6dx}{\sqrt{3-2x-x^2}}$ (12) $\int \frac{dx}{x^2-25}$

(13) $\int \frac{dx}{\sqrt{2x-x^2}}$ (14) $\int \frac{dx}{(tan^{-1} x)(1+x^2)}$ (15) $\int \frac{cos(sec^{-1} x)}{x\sqrt{x^2-1}}dx$

(16) $\int \frac{\sqrt{tan^{-1} x}dx}{1+x^2}$ (17) $\int \frac{(sin^{-1} x)^2}{\sqrt{1-x^2}}$ (18) $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$ (19) $\int \frac{e^{\cos^{-1} x}dx}{\sqrt{1-x^2}}$

(6) INVERSE HYPERBOLIC FUNCTIONS: **الدوال الزائدية العكسية:**

(i) $\int \frac{dx}{a^2-x^2} = \frac{1}{a}tan^{-1}\frac{x}{a} = \frac{1}{2a}ln\left|\frac{a+x}{a-x}\right|$ $,(a > |x|)$

(ii) $\int \frac{dx}{x^2-a^2} = \frac{1}{a}tan^{-1}\frac{x}{a} = \frac{1}{2a}ln\left|\frac{a-x}{a+x}\right|$ $,(a < |x|)$

(iii) $\int \frac{dx}{\sqrt{x^2-a^2}} = cosh^{-1}\frac{x}{a} + c = ln\left[x + \sqrt{x^2-a^2}\right] + c$

(iv) $\int \frac{dx}{\sqrt{x^2+a^2}} = sinh^{-1}\frac{x}{a} + c = ln\left[x \pm \sqrt{x^2-a^2}\right] + c$

(v) $\int \frac{xdx}{\sqrt{a^2+x^2}} = \sqrt{a^2+x^2} + c$

(vi) $\int \frac{xdx}{\sqrt{a^2-x^2}} =- \sqrt{a^2+x^2} + c$

(vii) $\int \frac{xdx}{\sqrt{a^2-x^2}} =-\frac{1}{a}sech^{-1}\frac{x}{a} + c$

(viii) $\int \frac{xdx}{\sqrt{a^2+x^2}} =-\frac{1}{a}csch^{-1}\frac{x}{a} + c$

Examples:

Evaluate the following integrals: احسب التكاملات الاتية:

(1) $\int \frac{dx}{\sqrt{4+x^2}}$ (2) $\int \frac{3dx}{x^2-16}$ (3) $\int \frac{dx}{\sqrt{x^2-25}}$ (4) $\int \left[\frac{3}{x}+\sqrt{x}\right] dx$

Solutions:

(1) $\int \frac{dx}{\sqrt{4+x^2}} = \sinh^{-1}\frac{x}{2} + c$ (2) $\int \frac{3dx}{x^2-16} = \frac{3}{4}\sinh^{-1}\frac{x}{4}$

(3) $\int \frac{dx}{\sqrt{x^2-25}} = \frac{1}{5}\sec^{-1}\frac{x}{5} + c$ (4) $\int \left[\frac{3}{x}+\sqrt{x}\right] dx = 3\int \frac{dx}{x} dx = 3lnx + \frac{2}{3}x^{\frac{3}{2}} + c$

Exercise:(6)

Evaluate the following integrals: احسب التكاملات الاتية:

(1) $\int \frac{1+sin^3x}{sin^2x} dx$ (2) $\int \frac{dx}{x\sqrt{1+(lnx)^2}}$ (3) $\int \frac{coscdx}{\sqrt{1+sin^2x}}$ (4) $\int \frac{dx}{1-x^2}$

(5) $\int \frac{dx}{x\sqrt{1-16x^2}}$ (6) $\int \frac{xdx}{x-1}$ (7) $\int \frac{xdx}{\sqrt{x}}$ (8) $\int \frac{dx}{\sqrt{x}}$

(B) Right, or wrong? Say which for each formula and gives a brief reason for each answer

وضح اي من الاجوبة الاتية صحيحة ومن ثم علل ذلك لكل اجابة

(1) $(i)\ \int tanx\, sec^x x\, dx = \frac{sec^3x}{3} + c$

$(ii)\ \int tanx\, sec^x x\, dx = \frac{1}{2}tan^2x + c$

$(iii)\ \int tanx\, sec^x x\, dx = \frac{1}{2}sec^2x + c$

(2) $(i)\ \int xsinx\, dx = \frac{x^2}{2}sinx + c$

$(ii)\ \int xsinx\, dx =- xcosx + c$

$(iii)\ \int xsinx\, dx =- xcosx + sinx + c$

(1) $(i)\ \int \frac{dx}{\sqrt{1+x^2}} dx = sin^{-1}x + c$

$(ii)\ \int \frac{dx}{\sqrt{1+x^2}} dx = sinh^{-1}x +$

$(iii)\ \int \frac{dx}{\sqrt{1+x^2}} dx = cosh^{-1}x + c$

(A) Verify the integration formula in Exercise 1-4 تحقق من التكاملات الاتية في التمارين من 1 الي 4

(1) $\int \frac{tan^{-1} x\,dx}{x^2} = lnx - \frac{1}{2} ln(1+x^2) - \frac{tan^{-1} x}{x} + c$

(2) $\int x^3 cos^{-1} 5x\,dx = \frac{x^4}{4} cos^{-1} 5x + \frac{5}{4}\int \frac{x^4 dx}{\sqrt{1-25x^2}}$

(3) $\int (sin^{-1} x)^2\,dx = x(sin^{-1} x)^2 - 2x + 2\sqrt{1-x^2}\, sin^{-1} x + c$

(4) $ln(a^2+x^2)dx = xln(a^2+x^2) - 2x + 2a\, tan^{-1}\frac{x}{a} + c$

SPECIAL METHODS OF INTEGRATION طرق خاصة للتكامل:

(1) INTEGRATION BY SUBISTITUTION التكامل بالتعويض: