

**(3) Trigonometric functions**

**الدوال المثلثية:**

$$(1) \int \sin x dx = -\cos x + c$$

$$(2) \int \cos x dx = \sin x + c$$

$$(3) \int \tan x dx = \ln|\sec x| = -\ln|\cos x| + c$$

$$(4) \int \cot x dx = \ln|\sin x| + c$$

$$(5) \int \sec x dx = \ln|\sec x + \tan x| + c$$

$$(6) \int \csc x dx = \ln|\csc x - \cot x| + c$$

$$(7) \int \sec^2 x dx = \tan x + c$$

$$(8) \int \csc^2 x dx = -\cot x + c$$

$$(9) \int \sec x \tan x dx = \sec x + c$$

$$(10) \int \csc x \cot x dx = -\csc x + c$$

**Examples:**

Evaluate the following integrals:

(احسب التكاملات الآتية)

$$(1) \int \sin 2x dx$$

$$(2) \int_0^{\frac{\pi}{4}} \cos 2\pi x dx$$

$$(3) \int \tan x dx$$

$$(4) \int \cot 3x dx$$

**Solutions:**

$$(1) \int \sin 2x dx = -\frac{1}{2} \cos 2x + c$$

$$(2) \int_0^{\frac{\pi}{4}} \cos 2\pi x dx = \frac{1}{2\pi} [\sin 2\pi x]_0^{\frac{\pi}{4}} + c = \frac{1}{2\pi} \left( \sin \frac{\pi}{2} - \sin 0 \right) = \frac{1}{2\pi} + c$$

$$(3) \int \tan x dx = -\ln|\cos x| + c$$

$$(4) \int \cot 3x dx = \frac{1}{3} \ln|\sin 3x| + c$$

**Example:**

Evaluate the following integrals:

(احسب التكاملات الآتية)

$$(1) \int_{\frac{\pi}{2}}^{\pi} \frac{4\cos x}{3+2\sin x} dx$$

$$(2) \int (\tan 2x + \sec x \tan x) dx$$

**Solution:**

$$(1) \int_{\frac{\pi}{2}}^{\pi} \frac{4\cos x}{3+2\sin x} dx = 2[\ln|3+2\sin x|]_{\frac{\pi}{2}}^{\pi} = 2\ln\left|\frac{-\pi}{2}\right| - 2\ln\left|\frac{\pi}{2}\right| = 2\ln|1| - 2\ln|5| = -2\ln 5$$

$$(2) \int (\tan 2x + \sec x \tan x) dx = \int \tan 2x dx + \int (\sec x \tan x) dx = \frac{1}{2} (\ln|\sec 2x|) + \sec x + c$$

**Examples:**

Calculate the following integrals: (احسب التكاملات الآتية)

$$(1) \int \sec 2x \, dx \quad (2) \int \csc^2 7x \, dx \quad (3) \int \csc 9x \, dx$$

**Solution:**

$$(1) \int \sec 2x \, dx = \frac{1}{2} \ln |\sec 2x + \tan 2x| + c$$

$$(2) \int \csc^2 7x \, dx = \frac{-1}{7} \cot 7x + c$$

$$(3) \int \csc 9x \, dx = \frac{1}{9} \ln |\csc 9x - \cot 9x| + c$$

**Example:**

Prove the following integrals: (احسب التكاملات الآتية)

$$(1) \int \tan x \, dx = -\ln \cos x \quad (2) \int \sec x \, dx = \ln |\sec x + \tan x|$$

**Solution:**

$$(1) \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{-\sin x}{\cos x} \, dx = -\ln \cos x$$

$$(2) \int \sec x \, dx = \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} \, dx = \ln |\sec x + \tan x|$$

**Exercise (2):**

(A) Find the simplest ant derivatives you can for each part and add the arbitrary constant of integration at the end.

احسب ابسط صورة ممكنة للتكاملات الآتية ثم اضع ثابت التكامل لكل حالة.

$$(1) \int (3x^2 + \sec x) \, dx \quad (2) \int (2x - \cos 7x + \sqrt{2x}) \, dx \quad (3) \int (\tan 6x + \sec^2 x) \, dx$$

$$(4) \int (5^{3x} + e^{-x} + x^{-1}) \, dx \quad (5) \int (4 - 8x)^9 \, dx \quad (6) \int \left( \frac{t\sqrt{t} + \sqrt{t}}{t^2} \right) dt$$

$$(7) \int \frac{\sec^2 x}{4} \, dx \quad (8) \int 5 \sec x \tan x \, dx \quad (9) \int 6 \cos 3x \, dx$$

$$(10) \int \left( x - \frac{1}{x} \right) \left( x + \frac{1}{x} \right) \, dx \quad (11) \int x^{-3} (x + 1) \, dx \quad (12) \int \frac{1}{2} \sqrt[5]{x} \, dx$$

- (13)  $\int (x-4)(x+2) dx$  (14)  $\int \sec^2 x \tan x dx$  (15)  $\int x^{-1} \ln x dx$   
 (16)  $\int x \sinh(x^2) dx$  (17)  $\int x \tanh(x^2 + 1) dx$  (18)  $\int \sqrt{1 - \cos^2 x} dx$   
 (19)  $\int \tan^2 x dx$  (20)  $\int \sin^3 x \cos x dx$  (21)  $\int \frac{e^x - 1}{e^x} dx$   
 (22)  $\int x e^{x^2} dx$  (23)  $\int e^x \cos x dx$  (24)  $\int \cos x \sin x dx$   
 (25)  $\int \frac{x^2}{x^2 + 4} dx$  (26)  $\int \sin^3 x \cos^5 x dx$  (27)  $\int \frac{(\sin x)^4}{\sqrt{1 - \cos^2 x}} dx$   
 (28)  $\int e^x \sin(e^x) dx$  (29)  $\int \frac{1}{x-4} dx$  (30)  $\int \ln x dx$

(B) **Prove that:**

- (1)  $\int \cot x dx = \ln |\sin x|$  (2)  $\int \tan x dx = \ln |\sec x|$   
 (3)  $\int \csc x dx = \ln |\csc x - \cot x|$

#### (4) HYPERBOLIC FUNCTIONS:

#### الدوال الزائدية

- (1)  $\int \sinh x dx = \cosh x + c$  (2)  $\int \cosh x dx = \sinh x + c$  (3)  $\int \tanh x dx = \ln |\cosh x| + c$   
 (4)  $\int \coth x dx = \ln |\sinh x| + c$  (5)  $\int \operatorname{csch}^2 x dx = -\coth x + c$  (6)  $\int \operatorname{sech}^2 x dx = \tanh x + c$   
 (7)  $\int \operatorname{sech} x \tanh x dx = -\operatorname{sech} x + c$  (8)  $\int \operatorname{csch} x \coth x dx = -\operatorname{csch} x + c$   
 (9)  $\int \operatorname{sech} x dx = \tanh^{-1}(\sinh x) + c$  (10)  $\int \operatorname{sech} x dx = \ln |\operatorname{sech} x + \tanh x| + c$   
 (11)  $\int \operatorname{csch} x dx = -\coth^{-1}(\cosh x) + c = \ln |\csc x - \cot x| + c$

#### Example:

Evaluate the integrals احسب التكاملات الآتية:

- (1)  $\int \sinh 8x dx$  (2)  $\int \operatorname{sech} 2x dx$  (3)  $\int \cosh^2 5x dx$  (4)  $\int \coth 3x dx$

#### Solution:

- (1)  $\int \sinh 8x dx = \frac{1}{8} \cosh 8x + c$  (2)  $\int \operatorname{sech} 2x dx = \frac{1}{2} \ln |\operatorname{sech} 2x + \tanh 2x| + c$   
 (3)  $\int \cosh^2 5x dx = \frac{-1}{5} \coth 5x + c$  (4)  $\int \coth 3x dx = \frac{1}{3} \ln |\sinh 3x| + c$

### Exercise:3

Evaluate the integrals:

احسب التكاملات الآتية:

- (1)  $\int \sinh \frac{x}{6} dx$  (2)  $\int \cosh \left( \frac{x}{2} - \ln 3 \right) dx$  (3)  $\int \frac{\operatorname{csch}(\ln x) \coth(\ln x)}{x} dx$
- (4)  $\int \cosh(3x - \ln 5) dx$  (5)  $\int 2 \sinh(\sin x) \cos x dx$  (6)  $\int \frac{\cosh \sqrt{x}}{\sqrt{x}} dx$
- (7)  $\int \frac{x}{\sqrt{x^2 - 2}} dx$  (8)  $\int \cosh(\tan x) \sec^2 x dx$  (9)  $\int \frac{1}{(2-x)^2} dx$
- (10)  $\int \sin 3x \cos 2x dx$  (11)  $\int x \sqrt{x^2 - 7} dx$  (12)  $\int \frac{\cos 3x}{4 + \sin 3x} dx$
- (13)  $\int \sin 2x \sin 3x dx$  (14)  $\int \frac{2x}{\sqrt{x^2 + 4}} dx$  (15)  $\int \frac{3x^2}{\sqrt{x^3 - 2}} dx$

### General Exercise:4

- (1)  $\int \frac{1 - \cos 3x}{5} dx$  (2)  $\int \frac{1}{2\sqrt{x} + 2x} dx$  (3)  $\int \frac{2 \ln x}{x} dx$  (4)  $\int \frac{1}{x(\ln x)^2} dx$
- (5)  $\int \frac{\sec x}{\sqrt{\ln(\sec x + \tan x)}} dx$  (6)  $\int \frac{\sec x \tan x}{2 + \sec x} dx$  (7)  $\int (\sin 3x - \csc^2 x) dx$
- (8)  $\int \sin^9 x dx$  (9)  $\int \frac{\sin x}{2 - \cos x} dx$  (10)  $\int \frac{\csc x}{\csc x - \sin x} dx$
- (11)  $\int (1 - \cot^2 x) dx$  (12)  $\int \frac{1}{1 + e^x} dx$  (13)  $\int e^x dx$
- (14)  $\int 2xe^{x^2} \cos(e^{x^2}) dx$  (15)  $\int \frac{e^x}{1 + e^{2x}} dx$  (16)  $\int (1 + e^{\cot x}) \csc^2 x dx$
- (17)  $\int 3^{-2 \cos x} \sin x dx$  (18)  $\int x \sin x^2 dx$  (19)  $\int e^{\sec x} \sec x \tan x dx$
- (20)  $\int x \ln e^x dx$  (21)  $\int 7^{\cos x} \sin x dx$  (22)  $\int \frac{1}{x^2} dx$

(B) Right or wrong? Say which for each formula and give a brief reason for each answer.

وضح اي من الاجوبة الآتية صحيحة ومن ثم علل ذلك في كل اجابة

(1)  $\int \tan x \sec^3 x dx = \frac{\sec^3 x}{3} + c$  (2)  $\int \tan x \sec^3 x dx = \frac{1}{2} \tan^2 x + c$

(2)  $\int \tan x \sec^3 x dx = \frac{1}{2} \sec^2 x + c$



$$(1) \int x \sin x^2 dx = \frac{x^2}{2} \sin x + c \quad (2) \int x \sin x^2 dx = -x \cos x^2 + c$$

$$(2) \int x \sin x^2 dx = \frac{-1}{2} \cos x^2 + c$$

## (5) INVERSE TRIGONOMETRIC FUNCTIONS:

## الدوال المثلثية العكسية

التكاملات الآتية دوال مثلثية عكسية وجميع النتائج صحيحة عند  $a \neq 0$

Integrals evaluated with inverse trigonometric functions the following formula hold for any constant  $a \neq 0$

$$(1) \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} = \cos^{-1} \frac{x}{a} \quad (\text{valid } x^2 < a^2)$$

$$(2) \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} = -\frac{1}{a} \cot^{-1} \frac{x}{a} \quad (\text{valid for all } x)$$

$$(3) \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} = -\frac{1}{a} \csc^{-1} \frac{x}{a} \quad (\text{valid } |x| > a > 0)$$

### Examples:

Evaluate the following integrals: احسب التكاملات الآتية :

$$(1) \int \frac{dx}{x^2 + 16} \quad (2) \int \frac{dx}{\sqrt{9 - x^2}} \quad (3) \int \frac{dx}{x\sqrt{x^2 - 4}} \quad (4) \int (\sin 3x + \frac{6}{x^2 + 5}) dx$$

$$(5) \int \frac{\sqrt{3}}{\sqrt{2}} \frac{dx}{\sqrt{1 - x^2}} \quad (6) \int_0^1 \frac{dx}{1 + x^2} \quad (7) \int \frac{dx}{\sqrt{3 - 4x^2}} \quad (8) \int \frac{\sqrt{2}}{\sqrt{3}} \frac{dx}{x\sqrt{x^2 - 1}}$$

### Solutions:

$$(1) \int \frac{dx}{x^2 + 16} = \frac{1}{4} \tan^{-1} \left( \frac{x}{4} \right) + c \quad (2) \int \frac{dx}{\sqrt{9 - x^2}} = \sin^{-1} \left( \frac{x}{3} \right) + c$$

$$(3) \int \frac{dx}{x\sqrt{x^2 - 4}} = -\frac{1}{2} \sec^{-1} \left( \frac{x}{2} \right) + c \quad (4) \int (\sin 3x + \frac{6}{x^2 + 5}) dx = -\frac{1}{3} \cos 3x + \frac{6}{\sqrt{5}} \tan^{-1} \left( \frac{x}{\sqrt{5}} \right) + c$$

$$(5) \int \frac{\sqrt{3}}{\sqrt{2}} \frac{dx}{\sqrt{1 - x^2}} = [\sin^{-1} x]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} = \sin^{-1} \left( \frac{\sqrt{3}}{2} \right) - \sin^{-1} \left( \frac{\sqrt{2}}{2} \right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

$$(6) \int_0^1 \frac{dx}{1 + x^2} = (\tan^{-1} x)_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$(7) \int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2} \int \frac{du}{\sqrt{(\sqrt{3})^2 - u^2}} = \frac{1}{2} \left[ \sin^{-1} \left( \frac{u}{\sqrt{3}} \right) \right] = \frac{1}{2} \sin^{-1} \left( \frac{2x}{\sqrt{3}} \right) + c \quad \text{Where}$$

$$a = \sqrt{3} \text{ and } u = 2x \rightarrow du = 2dx \rightarrow dx = \frac{du}{2}$$

$$(8) \int_{\frac{2}{\sqrt{3}}}^{\frac{\sqrt{2}}{2}} \frac{dx}{x\sqrt{x^2-1}} = [\sec^{-1} x]_{\frac{2}{\sqrt{3}}}^{\frac{\sqrt{2}}{2}} = \left( \frac{\pi}{4} \right) - \left( \frac{\pi}{6} \right) = \frac{\pi}{12}$$

### Excercise :

By completing the square evaluate

باكمال المربع احسب:

$$(1) \int \frac{dx}{\sqrt{4x-x^2}}$$

$$(2) \int \frac{dx}{x^2+4x+20}$$

$$(3) \int \frac{dx}{\sqrt{e^{2x}-6}}$$