جامعة العلوم والتقانة
University of Science and Technology

Faculty of Engineering

Calculus and it's Applications -1

Lecture - Three

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Types of Functions

1. Polynomial Functions :

$$f(x) = a_0 + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$$

Is called a polynomial function of degree n, where $a_0, a_1, a_2, \dots, a_n$ are constants

for example, $f(x) = 4x^3 - x^2 + 4x + 2$ polynomial of degree 2 . If $n = 0$

$f(x)$ is a constant function for example, $f(x) = k, \quad where\ k\ is\ const$

2. Rational Functions:

Are the functions in the form $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where $p(x)\ and\ q(x)$ are polynomial functions For example, $f(x) = \frac{x^2+x-2}{x-3}$

3. Exponential Functions

(i) $f(x) = a^x$, $a \equiv bas$, $(a > 0$, $a \neq 1)$ For example, $f(x) = 4^x$

(ii) $f(x) = e^x$, $e \approx 2.7182\ldots$ where e is called the natural base

Properties:

(1) $a^x a^y = a^{x+y}$ (2) $\frac{a^x}{a^y} = a^{x-y}$ (3) $(a^x)^y = a^{xy}$

(4) $a^{-x} = \frac{1}{a^x}$ (5) $a^x b^x = (ab)^x$ (6) $\frac{a^x}{b^x} = (\frac{a}{b})^x$

Example:

If $f(x) = 2^x$ prove that $f(x+3) - f(x-1) = \frac{15}{2} f(x)$

Solution

$$f(x+3) = 2^{x+3} = 2^x . 2^3 = 8(2^x)$$

$$f(x-1) = 2^{x-1} = 2^x\, 2^{-1}$$

$$f(x+3) - f(x-1) = 8(2^x) - 2^{-1}(2^x) = \frac{15}{2}(2^x)$$

$$f(x+3) - f(x-1) = \frac{15}{2} f(x)$$

4. Logarithmic Functions:

(i) $f(x) = log_a\, x$, $a \equiv base$ (ii) $f(x) = log_e\, x = ln\, x$

ln x is called the natural logarithm For example, $f(x) = log_e x$

Properties:

(1) $log(xy) = log\, x + log\, y$ (2) $log\,(\frac{x}{y}) = log\, x - log\, y$

(3) $log\, x^n = n\, log\, x$ (4) $log\,(\frac{1}{x}) = log\,(x)^{-1} = -\,log\, x$

(5) $log_a 1 = 0$ (6) $log_a a = 1$

(7) $ln\, e = 1$ (8) $ln\, e^x = x$ (9) $e^{ln\, x} = x \quad , x > 0$

5. Trigonometric Functions:

$f(x) = sin\, x$, $f(x) = cos\, x$, $f(x) = tan\, x$
$f(x) = cot\, x$, $f(x) = sec\, x$, $f(x) = csc\, x$ or $cosecx$

Laws:

(1) $sin^2 x + cos^2 x = 1$

(2) $1 + tan^2 x = sec^2 x$

(3) $sin\, 2x = 2\, sin\, x\, cos\, x$

(4) $cos\, 2x = 1 - 2\, sin^2 x = 2cos^2 x - 1 = cos^2 x - sin^2 x$

(5) $tan\, 2x = \frac{2tan\, x}{1-tan^2 x}$

(6) $sin(x \pm y) = sin\, x\, cos\, y \pm cos\, x\, sin\, y$

(7) $cos(x \pm y) = cos\, x\, cos\, y \mp sin\, x\, sin\, y$

(8) $sin\, x + sin\, y = 2\, sin\, \frac{x+y}{2}\, cos\, \frac{x-y}{2}$

(9) $sin\, x - sin\, y = 2\, cos\, \frac{x+y}{2}\, sin\, \frac{x-y}{2}$

(10) $cos\, x + cos\, y = 2\, cos\, \frac{x+y}{2}\, cos\, \frac{x-y}{2}$

(11) $cos\, x - cos\, y = 2\, sin \frac{x+y}{2}\, sin\, \frac{y-x}{2}$

6. Inverse Trigonometric Functions

$f(x) = \sin^{-1} x$, $f(x) = \cos^{-1} x$, $f(x) = \tan^{-1} x$
$f(x) = \cot^{-1} x$, $f(x) = \sec^{-1} x$, $f(x) = \csc^{-1} x$

If $y = \sin^{-1} x$ then $x = sin\, y$

$sin(\sin^{-1} x) = x\,,\ \sin^{-1}(sin\, x) = x$

7. Hyperbolic Functions:

$$f(x) = sinh\, x\,, \quad f(x) = cosh\, x\,,\ f(x) = tanh\, x$$
$$f(x) = coth\, x\,,\ f(x) = sech\, x\,,\ f(x) = csch\, x$$
$$sinh\, x = \frac{e^x - e^{-x}}{2}\ ,\ coshx = \frac{e^x + e^{-x}}{2}$$

$$tanh\, x = \frac{sinh\, x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}\ ,coth\, x = \frac{cosh\, x}{\sinh x} =- \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$sech\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}\quad ,\quad cschx = \frac{1}{sinhx} = \frac{2}{e^x + e^{-x}}$$

Laws:

(1) $cosh^2\, x - sinh^2\, x = 1$ (2) $sech^2 x = 1 - tanh^2 x$

(3) $csch^2 x = coth^2 x - 1$

8. Inverse Hyperbolic Functions

$$f(x) = sinh^{-1}\, x\,, \quad f(x) = \cosh^{-1} x\ , \quad f(x) = \tanh^{-1} x$$
$$f(x) = \coth^{-1} x\quad , f(x) = \text{sech}^{-1}\, x\ , \quad f(x) = \text{csch}^{-1}\, x$$

If $y = sinh^{-1}\, x$ then $x = sinh\, y$

$$sinh(sinh^{-1}\, x) = x\,, sinh^{-1}(sinh\, x) = x$$

9. Algebraic Functions

$$p_0\,(x)y^n + p_1(x)y^{n-1} + p_2(x)y^{n-2} + \cdots + p_n(x) = 0$$

is called algebraic Function, where $p_0\,(x), p_1\,(x), p_2\,(x), \dots, p_n\,(x)$ are polynomial functions

for example, $x^4y^3 + 2x^3y - 3\,x = 0$

Exercise -2-

1. If $f(x) = 5^x$ Find:

$(i) f(2)$ $\quad (ii) f(-2)$ $\quad (iii)\ f\left(\frac{1}{2}\right)$ $\quad (iv)\ \frac{f(x+1)}{f(x-2)}$)

2. Prove that $e^{\,ln\, x} = x$

3. Prove that $sinh\, x + cosh\, x = e^x$?

Domain of Definition

The set of values of x for which the values of y are determined by the rule $f(x)$ is called the domain of definition of the function, denoted by $D_{f\in(-\infty,+\infty)}$.

(1) If $f(x)$ is polynomial function then $D_f = \mathbb{R}$

(2) If $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ then $D_f = \mathbb{R} - \{q(x) = 0\}$

(3) If $f(x) = \sqrt{p(x)}$ then $D_f = p(x) \geq 0$

(4) If $f(x) = \frac{p(x)}{\sqrt{q(x)}}$ then $D_f = q(x) > 0$

(5) If $f(x) = ln(p(x))$ then $D_f = p(x) > 0$

(6) If $f(x)$ is Exponential function then $D_f = \mathbb{R}$

Example:

Find the domain of definition of the following functions

(1) $f(x) = \frac{x-4}{x-3}$ (2) $f(x) = x^4 - 7x + 2$ (3) $f(x) = \sqrt{x^2 - 4}$

(4) $f(x) = \frac{x-2}{\sqrt{2x-6}}$ (5) $f(x) = \frac{1}{x^2-2x}$ (6) $f(x) = ln(x + 5)$

Solution

(1) $f(x) = \frac{x-4}{x-3} \Rightarrow x - 3 \neq 0 \Rightarrow x \neq 3 \Rightarrow D_f = \{x \in \mathbb{R}: x \neq 3\}$

(2) $f(x) = x^4 - 7x + 5 \Rightarrow D_f = \mathbb{R}$ (polynomial)

(3) $f(x) = \sqrt{x^2 - 4} \Rightarrow x^2 - 4 \geq 0 \Rightarrow x^2 \geq 4 \Rightarrow$

$Df = \{x \in \mathbb{R}: x \geq \pm 2\}$

(4) $f(x) = \frac{x-2}{\sqrt{2x-6}} \Rightarrow 2x - 6 > 0 \Rightarrow 2x > 6 \Rightarrow D_f = \{x \in \mathbb{R}: x > 3\}$

(5) $f(x) = \frac{1}{x^2-2x} \Rightarrow x^2 - 2x \neq 0 \Rightarrow x(x-2) \neq 0 \Rightarrow$

$D_f = \{x \in \mathbb{R}: x \neq \{0,2\}\}$

(6) $f(x) = ln(x + 5) \Rightarrow x + 5 > 0 \Rightarrow x > -5 \Rightarrow$

$D_f = \{x \in \mathbb{R}: x > -5\}$

Even and Odd Functions

A function $f(x)$ is said to be an even if $f(-x) = f(x)$ and is said to be an odd if $f(-x) = -f(x)$

Example:

Which of the following functions are even and which are odd?

(1) $f(x) = x^3 + 2x$ (2) $f(x) = x^4 + 2x^2$

(3) $f(x) = \sin x$ (4) $f(x) = x \tan x$ (5) $f(x) = x^2 - 3x + 4$

Solution

(1) $f(x) = x^3 + 2x$

$f(-x) = (-x)^3 + 2(-x) - 3 = -x^3 - 2x = -(x^3 + 2x)$

$= -f(x)$

$\therefore -f(x)$ is an odd function

(2) $f(x) = x^4 + 2x^2$

$f(-x) = (-x)^4 + 2(-x)^2 = x^4 + 2x^2 = x^4 + 2x^2 = f(x)$

$\therefore f(x)$ is an even function

(3) $f(x) = \sin x$

$f(-x) = sin(-x) = sin(0 - x)$

$= sin\, 0 \cos x - sin\, x \cos 0 = 0 - sin\, x$

$= -sin\, x = -f(x)$

$\therefore f(x)$ is an odd function

(4) $f(x) = xtanx$

$$f(-x) = xtanx = \frac{x\, sinx}{cosx} = \frac{(-x)\sin(-x)}{\cos(-x)} = \frac{x\, sinx}{cosx} = f(x)$$

$\therefore f(x)$ is not odd and not even

Rule: $\sin(-x) = -sinx$, $\cos(-x) = cosx$

(5) $f(x) = x^2 - 3x + 4$

$f(-x) = (-x)^2 + 3(-x) + 4 = x^2 - 3x + 4 \neq f(x)$

$\therefore f(x)$ is not odd and not even function

The Limit of a Functions

Definition:

Suppose that $f(x)$ is defined when x is near the number a then we write

$\lim_{x \to a} f(x) = L$

and we say the limit of $f(x)$ as x approaches a equals L

Rule: $\lim_{x \to a} k = k$ *where k is const*

Properties:

1. $\mathrm{llim}_{x\to a} k = k$
2. $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x)$
3. $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
4. $\lim_{x\to a}(f(x).\ g(x)) = \lim_{x\to a} f(x).\ \lim_{x\to a} g(x)$
5. $\lim_{x\to a}(\frac{f(x)}{g(x)}) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
6. $\lim_{x\to a}(f(x))^n = (\lim_{x\to a} f(x))^n$

7. $\lim_{x\to a}(\ln f(x)) = \ln(\lim_{x\to a}(f(x))$

One Sided Limits

If $f(x)$ approaches L as x approaches a through values less than a $(x < a)$ then we say that L is a limit of $f(x)$ from the left and written
$\lim_{x\to \bar{a}} f(x) = L$

If $f(x)$ approaches L as x approaches a through values greater than a $(x > a)$ then we say that L is a limit of $f(x)$ from the right and written
$\lim_{x\to a^+} f(x) = L$

$f(x) = L$ if and only if $\lim_{x\to a} f(x) = L$

Rule : we say that the limite fund if $\lim_{x\to \bar{a}} f(x) = \lim_{x\to a^+} f(x)$

Example (1):

Find:

(1) $\lim_{x\to 0} x^2 + 4x - 5$ (2) $\lim_{x\to 1}\sqrt{2x+4}$ (3) $\lim_{x\to 0} \cos x + x^2$

(4) $\lim_{x\to 0} \ln(x+7)$ (5) $\lim_{x\to 3} \frac{x^2+1}{x-2}$ (6) $\lim_{x\to 0} 9$

Solution

(1) $\lim_{x\to 0} x^2 + 4x - 5 = 0^2 + 4(0) - 5 = -5$

(2) $\lim_{x\to 1}\sqrt{2x+4} = \sqrt{2+4} = \sqrt{6}$

(3) $\lim_{x\to 0} \cos x + x^2 = \cos 0 - 0^2 = 1 - 0 = 1$

(4) $\lim_{x\to 0} \ln(x+7) = \ln(0+7) = \ln 7$

(5) $\lim\limits_{x\to 3}\frac{x^2+1}{x-2}=\frac{9+1}{3-2}=10$

(6) $\lim\limits_{x\to 0}9=9$

Example (2):

If $f(x)=\begin{cases}2x-1\,, & x\geq 1\\ x^2\,, & x<2\end{cases}$ Find:

(1) $\lim\limits_{x\to 1^+}\mathrm{f}(x)$ (2) $\lim\limits_{x\to 1}\mathrm{f}(x)$ (3) $\lim\limits_{x\to 1}\mathrm{f}(x)$

Solution

(1) $\lim\limits_{x\to 1^+}\mathrm{f}(x)=\lim\limits_{x\to 1}2x-1=2-1=1$

(2) $\lim\limits_{x\to \bar{1}}\mathrm{f}(x)=\lim\limits_{x\to \bar{1}}\mathrm{x}^2=1^2=1$

(3) $\lim\limits_{x\to 1}\mathrm{f}(x)=1$ $\left(\lim\limits_{x\to 1^+}\mathrm{f}(x)=\lim\limits_{x\to \bar{1}}\mathrm{f}(x)\right)=1$

Exercise – 3

1. Find the domain of definition of the following functions

(1) $f(x)=x^2-3x+9$ (2) $f(x)=\ cotx-tanx$

(3) $f(x)\ =\ \sqrt{4-2x^2}$ (4) $f(x)\ =\ \ln(6-5x+x^2)$

(5) $f(x)=\ \sqrt{|2x\ -\ 5|}$ (6) $f(x)\ =\ \frac{2x}{x^{2-2x+1}}$

2. Which of the following functions are even and which are odd?

(1) $f(x)\ =\ x^6\ +\ 1$ (2) $f(x)=\ xcotx$ (3) $f(x)\ =\ x^2\ sec\ x$

(4) $f(x)=\frac{tanx}{x^2}$ (5) $f(x)=x^2-x+$ (6) $f(x)=sinhx$

Indeterminate Forms

Indeterminate forms

theorems or mathematical operations, for example factoring, multiplying by the

conjugate, dividing numerator and denominator by the highest power of x.

Note:

≡ any number

Example: